MATH 891: Topics on Nonlinear Waves

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Contents

1 Introduction

The linear wave equation is given by

$$\Box u(t,x) = \frac{\partial^2}{\partial t^2}u(t,x) - \Delta u(t,x)$$

where$(t,x) \in \mathbb{R}^{1+n}$ and $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$.
The defocusing energy critical wave equation in $\mathbb{R}^{1+3}$ (i.e. 3 spatial dimensions) with initial data is given by

$$\begin{cases} \Box u(t,x) = -u(t,x)^5 \\ u(0,\cdot) = f(\cdot), \quad \partial_t u(0,\cdot) = g(\cdot) \end{cases} \tag{1}$$

The specification of 3 spatial dimenstions is necessary for this equation to be energy critical. Reasons for this will be made clear in later sections.
In this topics course we will see a proof of the following main theorem:

Theorem 1.1. *Global existence for Defocusing Energy Critical Wave Equation*
Suppose $f, g \in C^\infty$ satisfy

$$\|\nabla f\|_{L^2} + \|g\|_{L^2} < \infty$$

Then a solution u of (1) *exists globally in time and $u \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^n)$.*

2 The Linear Equation

2.1 The Homogeneous Equation

In order to study the nonlinear equation, we first need to get well aquainted with the linear equation:

$$\begin{cases} \Box u = 0 \\ u(0,\cdot) = f; \quad \partial_t(0,\cdot) = g \end{cases} \tag{2}$$

where $u : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{C}$. This PDE is globally well-posed. That is, there exists a unique solution for all time for nice enough f and g.
Note on notation: The following notational conventions will be used:

$$\nabla u = \frac{\partial}{\partial x_j}u = \partial_j u, \quad (\partial_1 u, \ldots, \partial_n u), \quad \partial u = u' = (\partial_t u, \nabla u)$$

Theorem 2.1. *Homogeneous Wave Equation with Zero Initial Data*
Let $u \in C^2([0,T] \times \mathbb{R}^n)$ satisfy $\Box u = 0$. Fix $x_0 \in \mathbb{R}^n$ and $t_0 \in (0,T]$. Suppose $u = \partial_t u = 0$ for $t = 0$ and $|x - x_0| \leq t_0$.
Then $u \equiv 0$ in $\Omega = \{(t,x) : |x - x_0| \leq t_0 - t\}$

We can use theorem 2.1 to establish uniqueness for (2). If u and v both satisfy (2), then $w = u - v$ satisfies $\Box w = 0$ and $w(0,\cdot) = \partial_t w(0,\cdot) = 0$. Applying theorem 2.1 we see $u - v \equiv 0$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

Example 2.2. *Define* $\Phi(r) = \int_{B_r(x)} f(r,y)dy$. *This implies*

$$\Phi'(r) = \int_{B_r(x)} \partial_r f(r,y)dy + \int_{S(r,x)} f(r,y)d\sigma(y)$$

where $\sigma(y)$ *denotes the surface measure. The first term on the right hand side is obtained using the Dominated Convergence Theorem and the second term is obtained using the Fundamental Theorem of Calculus. See Selberg's notes for more detail.*

Proof. We will prove Theorem 2.1 using energy methods. Take $B_t = \{x : |x - x_0| \leq t_0 - t\}$. Thus B_t is a slice of the cone at height t. Now define

$$e(t) = \frac{1}{2}\int_{B_t} |u'(t,x)|^2 \; dx = \int_{B_t} (\partial_t u)^2 + \nabla u \cdot \nabla u \; dx$$

Heuristically, this is analogous to $\frac{1}{2}mv^2$ from introductory physics. We now calculate

$$\begin{aligned}
e'(t) &= \int_{B_t} u_t u_{tt} + \nabla u_t \cdot \nabla u \; dx - \frac{1}{2}\int_{\partial B_t} |u'|^2 \; d\sigma(x) \\
&= \int_{B_t} u_t u_t t - u_t \Delta u + u_t \Delta u + \nabla u_t \cdot \nabla u \; dx - \frac{1}{2}\int_{\partial B_t} |u'|^2 \; d\sigma(x) \\
&= \int_{B_t} u_t u_t t - u_t \Delta u + div(u_t \nabla u) \; dx - \frac{1}{2}\int_{\partial B_t} |u'|^2 \; d\sigma(x) \\
&= \int_{\partial B_t} u_t \nabla u \cdot \overrightarrow{n} \; d\sigma(yx) - \frac{1}{2}\int_{\partial B_t} |u'|^2 \; d\sigma(x)
\end{aligned}$$

Next we note

$$|u_t \nabla u \cdot \overrightarrow{n}| \leq |u_t||\nabla u||\overrightarrow{n}| \leq \frac{1}{2}\left(|u_t|^2 + |\nabla u|^2\right) = \frac{1}{2}|u'|^2$$

so that $e'(t) \leq 0$ for all t. This implies $e(t) \leq 0$ since $e(0) = 0$. On the other hand, $e(t) \geq 0$ for all t by definition. Thus $e(t) \equiv 0$. It follows that $u' \equiv 0$ in Ω and therefore $u \equiv 0$ in Ω. $\square$

Corollary 2.3. *Finite Speed of Propagation*
Let $u \in C^2$ *with* $\Box u = 0$ *and* $u(0,x) = \partial_t u(0,x) = 0$ *if* $|x| > R$. *Then we have* $u(t,x) = 0$ *for* $|x| > t + R$ *and* $0 \leq t \leq T$.

Theorem 2.4. *Huygens' Principle*
Suppose the assumptions of the above corollary hold. If $n \geq 3$ *and* n *is odd. Then* $u(t,x) = 0$ *unless* $t - R \leq |x| \leq t + R$.

This theorem is extremely easy to break. For example, by introducing geometry, boundary, or nonlinearity. Conservation of energy is a much more robust tool.

Theorem 2.5. *Conservation of Energy*
Set

$$E[u](t) = \frac{1}{2}\int_{\mathbb{R}^n} |u'(t,x)|^2 \; dx = \frac{1}{2}\|u'(t,\cdot)\|_{L^2}^2$$

. *If* $u \in C^2$ *and* $\Box u = 0$, *and for all* t *we have* $u(t,x) = 0$ *if* $|x|$ *sufficiently large, then* $E[u](t) = E[u](0)$ *for all* $0 \leq t \leq T$.

Proof.
$$\frac{d}{dt}E[u](t) = \int u_t u_{tt} + \nabla u \cdot \nabla u_t \; dx = \int u_t(\Box u) \; dx = 0$$

□

2.2 The Inhomogeneous Equation

Thus far we have discussed the homogeneous linear equation. We now turn our attention to the inhomogeneous equation, which is given by:

$$\begin{cases} \Box u = F \\ u(o, \cdot) = f; \quad \partial_t u(0, \cdot) = g \end{cases} \tag{3}$$

Note that if $u = v + w$ such that v satisfies $\Box v = 0; v(0, \cdot) = f; \partial_t v(0, \cdot) = g$ and w satisfies $\Box w = F; w(0, \cdot) \equiv 0 \equiv w_t(0, \cdot)$, then u satisfies (3).
Exercise: Assume F is "nice enough". For each s, let

$$\begin{cases} \Box v_s(t, x) = 0 \\ v_s(0, x) = 0 \quad \partial_t v_s(0, x) = F(s, x) \end{cases}$$

Then if $w(t, x) = \int_0^t v_s(t - s, x) \; ds$ then $\Box w = F; w(0, \cdot) \equiv 0 \equiv \partial_t w(0, \cdot)$.

Proposition 2.6. *Energy Boundedness*
Suppose u satisfies (3). *Then we have*

$$\|u'(t, \cdot)\|_{L^2} \lesssim \|u'(0, \cdot)\|_{L^2} + \int_0^t \|F(s, \cdot)\|_{L^2} \; ds$$

Proof. WLOG, take $f, g = 0$. Write $u(t, x) = \int_0^t v_s(t - s, x) \; ds$ where

$$\Box v_s = 0; \quad v_s(0, \cdot) = 0; \quad \partial_t v_s(0, \cdot) = F(s, \cdot).$$

We have
$$u'(t, x) = \int_0^t v_s'(t - s, \cdot) \; ds$$

so that

$$\begin{aligned} \|u'(t, \cdot)\|_{L^2} &\le \int_0^t \|v_s'(t - s, \cdot)\|_{L^2} \; dx \quad \text{by the Minkowski integral inequality} \\ &\lesssim \int_0^t \|F(s, \cdot)\|_{L^2} \; ds \end{aligned}$$

□

Proof. Alternate Proof
We have

$$\frac{d}{dt}E[u](t) = \int u_t \Box u \; dx = \int u_t F \; dx \leq \|u_t\|_{L^2} \|F(t\cdot)\|_{L^2} \lesssim \|u'(t,\cdot)\|_{L^2} \|F(t,\cdot)\|_{L^2}$$

We can also write the LHS:

$$\frac{d}{dt}E[u](t) = \frac{d}{dt}\frac{1}{2}\|u'(t,\cdot)\|_{L^2}^2 = \|u'(t,\cdot)\|_{L^2} \frac{d}{dt}\|u'(s,\cdot)\|_{L^2}$$

Combining the lines above we see

$$\frac{d}{dt}\|u'(s,\cdot)\|_{L^2} \lesssim \|F(s,\cdot)\|_{L^2} \Rightarrow \|u'(s,\cdot)\|_{L^2} \lesssim \int_0^t \|F(s,\cdot)\|_{L^2} \; ds$$

□

2.3 Solutions Via Fourier

Let $f \in L^1$. Then the Fourier transform of f is given by

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) = \frac{1}{2\pi^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) \; dx$$

We define the Schwartz class of functions as

$$\mathcal{S} = \{f \in C^\infty : \sup_x (1+|x|)^N |\partial^\alpha f| < \infty \; \forall N, \alpha\}$$

It is a fact that $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ isomorphically. We define the inverse Fourier transform:

$$\mathcal{F}^{-1}[f](x) = \frac{1}{2\pi^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi) \; d\xi$$

We also note the fact that $\mathcal{F}$ extends to L^2 as a unitary isomorphism.
The following properties are stated without proof. These can be easily verified using Fubini's Theorem and Integration By Parts

1. $\int \hat{f} g = \int f \hat{g}$
2. $f * g = \int f(x-y) g(y) dy \Rightarrow \mathcal{F}[f * g] = \hat{f}\hat{g}$
3. $\mathcal{F}[\partial^\alpha f](\xi) = (i\xi)^\alpha \hat{f}(\xi)$
4. $\partial_\xi^\alpha \hat{f}(\xi) = \mathcal{F}[(-ix)^\alpha f]$
5. $\|f\|_{L_x^2} = \|\hat{f}\|_{L_\xi^2}$ (Plancherel's Theorem)

Note that by 3, we have

$$(\Delta f)\hat{}(\xi) = -\sum_{j=1}^{n} \xi_j^2 \hat{f}(\xi) = -|\xi|^2 \hat{f}(\xi)$$

We use this to define $\sqrt{-\Delta} f = \mathcal{F}^{-1}[(|\xi|\hat{f})]$
Consider the homogeneous wave equation (2). Taking the Fourier transforms in space we obtain:

$$\mathcal{F}[(\partial_t^2 - \Delta)u] = \hat{0} = 0; \quad \hat{u}(0,\cdot) = \hat{f}; \quad \widehat{\partial_t u}(0,\cdot) = \hat{g}$$

Also note $\mathcal{F}[(\partial_t^2 - \Delta)u] = (\partial_t^2 + |\xi|^2)\hat{u}$. This gives

$$\hat{u}(t,\xi) = \hat{f}(\xi)\cos t|\xi| + \hat{g}(\xi)\frac{\sin t|\xi|}{|\xi|}$$

Now consider the inhomogeneous wave equation (3). Duhamel's principle gives

$$\hat{u}(t,\xi) = \hat{f}(\xi)\cos t|\xi| + \hat{g}(\xi)\frac{\sin t|\xi|}{|\xi|} + \int_0^t \frac{\sin(t-s)|\xi|}{|\xi|}\hat{F}(s,\xi)\ ds$$

Inverting the Fourier transform in space gives a formula for the solution $u(t,x)$:

$$\begin{aligned} u(t,x) = c_n & \iint e^{i(x-y)\cdot\xi} \cos t|\xi| f(y)\ dyd\xi \\ &+ c_n \iint e^{i(x-y)\cdot\xi}\frac{\sin t|\xi|}{|\xi|} g(y)\ dyd\xi \\ &+ c_n \int_0^t \iint e^{i(x-y)\cdot\xi}\frac{\sin(t-s)|\xi|}{|\xi|} F(x,y)\ dyd\xi ds \end{aligned}$$

2.4 Sobolev Spaces

Let $s \in \mathbb{N}$. Then we define the H^s norm by:

$$\|f\|_{H^s} = \left(\sum_{|\alpha|\le s} \|\partial^\alpha f\|_{L^2}^2\right)^{\frac{1}{2}} = \left(\sum_{|\alpha|\le s} \|\xi^\alpha \hat{f}(\xi)\|_{L^2}^2\right)^{\frac{1}{2}}$$

Note that $\sum_{|\alpha|\le s} |\xi^\alpha| \approx (1+|\xi|^2)^{s/2}$ so that $\|f\|_{H^s} \approx \|(1+|\xi|^2)^{s/2}\hat{f}(\xi)\|_{L^2}$. This motivates the following definition for $s \in \mathbb{R}$:

$$\|f\|_{H^s} = \|(1+|\xi|^2)^{s/2}\hat{f}(\xi)\|_{L^2}$$

It follows that if $s < t$ then $\|f\|_{H^s} \le \|f\|_{H^t}$ so that $H^t \subset H^s$.

Proposition 2.7. *A first Sobolev embedding*
If $s > \frac{n}{2}$ then $H^s \subset L^\infty$.

Note that one can construct a counterexample for $s \le \frac{n}{2}$.

Proof. It suffices to show $\|f\|_{L^\infty} \le C\|f\|_{H^s}$ if $s > \frac{n}{2}$.

$$\begin{aligned}
|f(x)| &= c_n \left| \int e^{ix\cdot\xi}\hat{f}(\xi)\ d\xi \right| \\
&\lesssim \int |\hat{f}(\xi)|\ d\xi \\
&= \int |(1+|\xi|^2)^{-s/2}(1+|\xi|^2)^{s/2}\hat{f}(\xi)|\ d\xi \\
&\le \left[\int (1+|\xi|^2)^{-s}\ d\xi\right]^{1/2} \|(1+|\xi|^2)^{s/2}\hat{f}(\xi)\|_{L^2}
\end{aligned}$$

The term in brackets is finite provided $2s > n$. Thus $\|f\|_{L^\infty} \le C\|f\|_{H^s}$ if $s > \frac{n}{2}$, as desired. □

We now discuss Homogeneous Sobolev Spaces. Define the $\dot{H}^s$ norm by

$$\|f\|_{\dot{H}^s} = \||\xi|^s\hat{f}\|_{L^2}$$

If $s \in \mathbb{N}$, then $\|f\|_{\dot{H}^s} \approx \left(\sum_{|\alpha|=s} \|\partial^\alpha f\|_{L^2}\right)^{1/2}$
This allows us to show the following scaling property:

$$\|f(\lambda x)\|_{\dot{H}^s} = \lambda^{s-\frac{n}{2}}\|f\|_{\dot{H}^s}$$

Proof.

$$[f(\lambda x)]\hat{}(\xi) = c_n \int e^{-ix\cdot\xi}f(\lambda x)\ dx = c_n\lambda^{-n}\int e^{-i\frac{x}{\lambda}\cdot\xi}f(x)\ dx = \lambda^{-n}\hat{f}\left(\frac{\xi}{\lambda}\right)$$

$$\begin{aligned}
\|f(\lambda x)\|_{\dot{H}^s}^2 &= \int |\xi|^{2s}|[f(\lambda x)]\hat{}(\xi)|^2\ d\xi \\
&= \lambda^{-2n}\int |\xi|^{2s}|\hat{f}\left(\frac{\xi}{\lambda}\right)|^2\ d\xi \\
&= \lambda^{-n}\int |\lambda\xi|^{2s}|\hat{f}(\xi)|^2\ d\xi \\
&= \lambda^{-n+2s}\|f\|_{\dot{H}^s}^2
\end{aligned}$$

□

<u>Exercise</u>: If f is compactly supported and $0 < s < \frac{n}{2}$, show $\|f\|_{H^s} \approx \|f\|_{\dot{H}^s}$

2.5 L^p Spaces

Here we have a quick review of some basic facts about L^p spaces. Define the L^p norm by

$$\|f\|_{L^p} = \left(\int |f|^p\ dx\right)^{1/p}$$

Then if $1 \le p < \infty$, L^p is a Banach space (a normed vector space).

Recall: Hölder's Inequality

$$\|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q}, \qquad p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1$$

This inequality can be further generalized:

$$\|fg\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}, \qquad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$$

2.6 An Introduction to Strichartz Estimates

Strichartz estimates area family of mixed norm estimates. We will be building up to proving some Strichartz estimates. Here we give an example of one such estimate and an application.
Let $u \in \mathbb{R} \times \mathbb{R}^3$ satisfy (3). Then the following estimate holds:

$$\|u\|_{L_t^4 L_x^4} \lesssim \|f\|_{\dot{H}^{1/2}} + \|g\|_{\dot{H}^{-1/2}} + \|\Box u\|_{L_t^{4/3} L_x^{4/3}}$$

The norm on the LHS is defined by

$$\|u\|_{L_t^4 L_x^4} = \|\|u\|_{L_x^4}\|_{L_t^4} = \left[\int_{\mathbb{R}} \int_{\mathbb{R}^3} |u(t,x)|^4 \, dx \, dt\right]^{1/4}$$

We will use this estimate to show global existence for

$$\Box u = u^3; \qquad u(0,\cdot) = f; \qquad \partial_t u(0,\cdot) = g$$

provided $\|f\|_{\dot{H}^{1/2}} + \|g\|_{\dot{H}^{-1/2}} = \epsilon << 1$ is sufficiently small.

Proof. We will prove this via iteration. Set $u_0 \equiv 0$ and define u_j to solve

$$\Box u_j = (u_{j-1})^3 \qquad u_j(0,\cdot) = f \qquad \partial_t u_j(0,\cdot) = g$$

We will show $\{u_j\}$ converges. See proof of Theorem 22 in Selberg for details that convergence implies it converges to u as above.

1. Inductively show there exists a uniform C_1 such that

$$\|u_j\|_{L_t^4 L_x^4} \leq 2C_1\epsilon$$

The Strichartz estimate implies $\|u_1\|_{L_t^4 L_x^4} \leq C_1(\|f\|_{\dot{H}^{1/2}} + \|g\|_{\dot{H}^{-1/2}}) = C_1\epsilon$. Assume $\|u_{j-1}\|_{L_t^4 L_x^4} \leq 2C_1\epsilon$. Then Stricharz implies

$$\begin{aligned}
\|u_j\|_{L_t^4 L_x^4} &\leq C_1\epsilon + C_1\|u_{j-1}^3\|_{L_t^{4/3} L_x^{4/3}} \\
&= C_1\epsilon + C_1\|u_{j-1}\|^3_{L_t^4 L_x^4} \\
&\leq C_1\epsilon + C_1(2C_1\epsilon)^3 \\
&\leq 2C_1\epsilon \qquad \text{provided } 8C_1^3\epsilon^2 \leq 1
\end{aligned}$$

2. Show $\{u_j\}$ is Cauchy.

   It suffices to show

$$\|u_j - u_{j-1}\|_{L^4_t L^4_x} \leq C\|u_{j-1} - u_{j-2}\|_{L^4_t L^4_x}$$

   Note that $(u_j - u_{j-1})$ solves

$$\Box(u_j - u_{j-1}) = u^3_{j-1} - u^3_{j-2} \qquad (u_j - u_{j-1})(0,\cdot) = \partial_t(u_j - u_{j-1})(0,\cdot) = 0$$

   Thus $\Box(u_j - u_{j-1}) = (u^2_{j-1} + u_{j-1}u_{j-2} + u^2_{j-2})(u_{j-1} - u_{j-2})$ and we note that $(u^2_{j-1} + u_{j-1}u_{j-2} + u^2_{j-2}) \in O(|u_{j-1}|^2 + |u_{j-2}|^2)$ Then Strichartz implies

$$\begin{aligned}\|u_j - u_{j-1}\|_{L^4L^4} &\leq C\|(|u_{j-1}|^2 + |u_{j-2}|^2)|u_{j-1} - u_{j-2}|\|_{L^{4/3}L^{4/3}} \\ &\leq C\left[\|u_{j-1}\|^2_{L^4L^4} + \|u_{j-2}\|^2_{L^4L^4}\right]\|u_{j-1} - u_{j-2}\|_{L^4L^4} \quad \text{by Hölder} \\ &\leq 2C(2C_1\epsilon)^2\|u_{j-1} - u_{j-2}\|_{L^4L^4}\end{aligned}$$

□

2.7 Building Background Tools

Theorem 2.8. *Riesz-Thorin Interpolation*
Let $T : L^{p_0} \cap L^{p_1} \to L^{q_0} \cap L^{q_1}$ be linear such that

$$\|Tf\|_{L^{q_j}} \leq M_j\|f\|_{L^{p_j}} \quad with 1 \leq p_j, q_j \leq \infty$$

Then if $0 < t < 1$, $\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}$, and $\frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$, then

$$\|Tf\|_{L^{q_t}} \leq M_0^{1-t}M_1^t\|f\|_{L^{p_t}}$$

Theorem 2.9. *Young's Inequality*
Let $Tf = \int K(x,y)f(y)\,dy$ and suppose $r \geq 1$. Furthermore, suppose

$$\sup_x \|K(x,\cdot)\|_{L^r}, \sup_y \|K(\cdot,y)\|_{L^r} \leq C.$$

If $1 \leq p \leq q \leq \infty$ and $\frac{1}{r} = 1 - \left(\frac{1}{p} - \frac{1}{q}\right)$. Then

$$T : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \quad and \quad \|Tf\|_{L^q} \leq C\|f\|_{L^p}$$

Proof. Take $\frac{1}{r} + \frac{1}{r'} = 1$. Suppose we can show

1. $\|Tf\|_{L^\infty} \leq C\|f\|_{L^{r'}}$
2. $\|Tf\|_{L^r} \leq C\|f\|_{L^1}$

If so, then

$$\left.\begin{aligned}\frac{1}{q} = \frac{1-t}{\infty} + \frac{t}{r} = \frac{t}{r} \\ \frac{1}{p} = \frac{1-t}{r'} + \frac{t}{1}\end{aligned}\right\} \Rightarrow \frac{1}{r} = 1 - \left(\frac{1}{p} - \frac{1}{q}\right)$$

So by Riesz-Thorin interpolation theorem, $\|Tf\|_{L^q} \leq C\|f\|_{L^p}$ We now prove

1.

$$\begin{aligned}\|Tf\|_{L^\infty} &= \left\|\int K(x,y)f(y)\ dy\right\|_{L^\infty_x}\\ &\le \|\|K(x,\cdot)\|_{L^r}\|f\|_{L^{r'}}\|_{L^\infty_x}\\ &\le C\|f\|_{L^{r'}}\end{aligned}$$

2.

$$\begin{aligned}\|Tf\|_{L^r} &= \|\int K(x,y)f(y)\ dy\|_{L^r}\\ &\le \int \|K(\cdot,y)\|_{L^r}|f(y)|\ dy \quad \text{by Minkowski integral inequality}\\ &\le \sup_y \|K(\cdot,y)\|_{L^r}\int |f(y)|\ dy\\ &\le C\|f\|_{L^1}\end{aligned}$$

□

Theorem 2.10. *Hardy-Littlewood Fractional Integral Inequality*
Suppose $0<\alpha<1$, $1<p<q<\infty$, *and* $\alpha = 1-\left(\frac{1}{p}-\frac{1}{q}\right)$. *Set* $I_\alpha f = \int_{-\infty}^{\infty} f(s)|t-s|^{-\alpha}\ ds$. *Then*

$$\|I_\alpha f\|_{L^q} \le C_{p,q,\alpha}\|f\|_{L^p}$$

Note: By Young's if $1+\frac{1}{q} = \frac{1}{p}+\frac{1}{r}$

$$\|I_\alpha f\|_{L^q} \le \|f\|_{L^p}\||s|^{-\alpha}\|_{L^r}$$

$$\||s|^{-\alpha}\|_{L^r}^r = \int_{-\infty}^{\infty}|s|^{-\alpha r}\ ds = \int_{-\infty}^{\infty}|s|^{-1}\ ds$$

This just barely fails...
Note: This cannot be true for any other α by scaling. To see this suppose not. That is, suppose $1<p<q<\infty, \beta\ne\alpha$, $0<\beta<1$, and

$$\|I_\beta f\|_{L^q} \le C\|f\|_{L^p} \quad \text{for all } f$$

We can take C to be the optimal constant. Set $f_\lambda(t) = f(\lambda t)$.

$$\begin{aligned}I_\beta f_\lambda(t) &= \int_{-\infty}^{\infty}|t-s|^{-\beta}f(\lambda s)\ ds\\ &= \lambda^{-1}\int_{-\infty}^{\infty}|t-\frac{s}{\lambda}|^{-\beta}f(x)\ dx\\ &= \lambda^{-1+\beta}\int_{-\infty}^{\infty}|\lambda t-s|^{-\beta}f(s)\ ds\\ &= \lambda^{-1+\beta}(I_\beta f)(\lambda t))\end{aligned}$$

Since C is optimal, there exists $\tilde{f}$ such that

$$\|I_\beta \tilde{f}\|_{L^q} \geq \frac{C}{2}\|\tilde{f}\|_{L^p}$$

But,

$$\|I_\beta \tilde{f}_\lambda\|_{L^q} \leq C\|\tilde{f}_\lambda\|_{L^p} = C\|\tilde{f}(\lambda\cdot)\|_{L^p} = C\lambda^{-1/p}\|\tilde{f}\|_{L^p}$$

$$\|I_\beta \tilde{f}_\lambda\|_{L^q} = \lambda^{-1+\beta}\|(I_\beta \tilde{f})(\lambda\cdot)\|_{L^q} = \lambda^{-1+\beta-\frac{1}{q}}\|I_\beta \tilde{f}\|_{L^q} \geq \lambda^{-1+\beta-\frac{1}{q}}\frac{C}{2}\|\tilde{f}\|_{L^p}$$

Thus we have $\frac{1}{2} \leq \lambda^{-\frac{1}{p}+1+\frac{1}{q}-\beta} = \lambda^{\alpha-\beta}$. Since λ is arbitrary, this must hold for all λ. Therefore $\alpha - \beta = 0$.

Theorem 2.11. *Hardy-Littlewood Maximal Inequality (n=1)*
Let

$$(\mathcal{M}f)(t) = \sup_r \frac{1}{2r}\int_{t-r}^{t+r} f(s)\ ds$$

Then if $1 < p \leq \infty$, $\|\mathcal{M}f\|_{L^p} \lesssim \|f\|_{L^p}$.

Proof. of Hardy-Littlewood Maximal Inequality

$$\begin{aligned}
|I_\alpha f(t)| &= \left|\int_{-\infty}^{\infty} f(s)|t-s|^{-\alpha}\ ds\right| \\
&\leq \int_{-\infty}^{\infty} |f(t-s)||s|^{-\alpha}\ ds \\
&= \int_{|s|<R} |f(t-s)||s|^{-\alpha}\ ds + \int_{|s|>R} |f(t-s)||s|^{-\alpha}\ ds
\end{aligned}$$

$$\begin{aligned}
\int_{R2^{-k}<|s|<R2^{-k+1}} |f(t-s)||s|^{-\alpha}\ ds &\leq (R2^{-k})^{-\alpha}\int_{|s|<R2^{-k+1}} |f(t-s)|\ ds \\
&= (R2^{-k})^{-\alpha}\int_{t-R2^{k+1}}^{t+R2^{k+1}} |f(s)|\ ds \\
&\leq (R2^{-k})^{-\alpha} 2R2^{-k+1}(\mathcal{M}|f|)(t)
\end{aligned}$$

$$\begin{aligned}
\int_{|s|<R} |f(t-s)||s|^{-\alpha}\ ds &= \sum_{k=1}^{\infty}\int_{R2^{-k}<|s|<R2^{-k+1}} |f(t-s)||s|^{-\alpha}\ ds \\
&\leq (\mathcal{M}f)(t)\sum_{k=1}^{\infty} R^{1-\alpha}42^{k(\alpha-1)} \\
&\leq CR^{1-\alpha}(\mathcal{M}f)(t) \\
&= CR^{\frac{1}{p}-\frac{1}{q}}(\mathcal{M}f)(t)
\end{aligned}$$

$$\begin{aligned}
\int_{|s|>R} |f(t-s)||s|^{-\alpha}\, ds &\leq \|f(t-\cdot)\|_{L^p} \left[\int_{|s|>R} |s|^{-\alpha p'}\, ds\right]^{1/p'} \\
&\leq \|f\|_{L^p} \left[2\int_R^\infty s^{-\alpha\frac{p}{p-1}}\, dp\right]^{\frac{p-1}{p}} \qquad \text{note: } \alpha\frac{p}{p-1} = 1 + \frac{1}{q}\frac{p}{p-1} > 1 \\
&= C\|f\|_{L^p} \left[R^{-\alpha\frac{p}{p-1}+1}\right]^{\frac{p-1}{p}} \\
&= CR^{-1/q}\|f\|_{L^p}
\end{aligned}$$

so we have

$$\int_{|s|<R} |f(t-s)||s|^{-\alpha}\, ds + \int_{|s|>R} |f(t-s)||s|^{-\alpha}\, ds \leq CR^{\frac{1}{p}-\frac{1}{q}}(\mathcal{M}f)(t) + CR^{-\frac{1}{q}}\|f\|_{L^p}$$

Choose $R = \frac{\|f\|_{L^p}^p}{|\mathcal{M}f(t)|^p}$, then we have

$$|I_\alpha f(t)| \leq 2C\|f\|_{L^p}^{1-\frac{p}{q}}(\mathcal{M}|f|)^{\frac{p}{q}}$$

Which implies

$$\begin{aligned}
\|I_\alpha f\|_{L^q} &\leq 2C\|f\|_{L^p}^{1-\frac{p}{q}}\|\mathcal{M}f^{\frac{p}{q}}\|_{L^q} \\
&= C\|f\|_{L^p}^{1-\frac{P}{q}}\|\mathcal{M}f\|_{L^p}^{\frac{p}{q}} \\
&\leq C\|f\|_{L^p}
\end{aligned}$$

□

2.7.1 Littlewood-Paley Theory

Fix $\chi \in C_c^\infty$ with $supp\chi \subseteq [-1,1]$ and so that $\chi \equiv 1$ on $[-\frac{1}{2},\frac{1}{2}]$. Define S_j via $\hat{S_j f} = \chi\left(\frac{|\xi|}{2^j}\right)\hat{f}(\xi)$ so that $S_j = \mathcal{F}^{-1}\xi\left(\frac{|\xi|}{2^j}\right)\mathcal{F}$. Set $\Delta_0 = S_0$ and $\Delta_j = S_j - S_{j-1}$. This is referred to as the Littlewood-Paley decomposition.
<u>claim</u>: $f = \sum_{j=0}^\infty \Delta_j f$. To see this we note

$$\sum_0^N \Delta_j f = S_N f = \mathcal{F}^{-1}\left(\xi\left(\frac{|\xi|}{2^N}\right)\hat{f}\right) \to f \text{ as } N \to \infty$$

Note that

$$\hat{\Delta_j f} = \beta_j(\xi)\hat{f}(\xi) \quad \text{with} \quad \beta_0(\xi) = \chi(\|\xi|) \text{ and } \beta_j(\xi) = \chi\left(\frac{|\xi|}{2^j}\right) - \chi\left(\frac{|\xi|}{2^{j-1}}\right)$$

Then β_j satisfies $0 \leq \beta_j \leq 1$ and $\sum_0^\infty \beta_j^2 \leq \sum_0^\infty \beta_j = 1$.
Properties

1. supp $\hat{S_j f} \subseteq \{|\xi| \leq 2^j\}$

2. supp $\hat{\Delta_j f} \subseteq \{2^{j-2} \leq |\xi| \leq 2^j\}$

3. $S_j f = 2^{jn}\Psi(2^j \cdot) * f$ such that $\hat{\Psi} = \chi$

4. $\Delta_j f = 2^j n\Phi(2^j \cdot) * f$ such that $\hat{\Phi} = \chi(\zeta) - \chi(2\zeta)$

We also have

$$p \geq 1 \Rightarrow \|S_j f\|_{L^p} \leq \|\Psi\|_{L^1}\|f\|_{L^p} \text{ and } \|\Delta_j f\|_{L^p} \leq \|\Phi\|_{L^1}\|f\|_{L^p}$$

Theorem 2.12. *If $1 < p < \infty$ then we have*

$$\|f\|_{L^p} \lesssim \left\| \left(\sum_j (\Delta_j f)^2 \right)^{1/2} \right\|_{L^p}$$

Proof. omitted. □

Corollary 2.13. *If $2 \leq p < \infty$ then*

$$\|f\|_{L^p} \lesssim \left(\sum_j \|\Delta_j f\|_{L^p}^2 \right)^{1/2}$$

And if $1 < q \leq 2$ then

$$\left(\sum_j \|\Delta_j f\|_{L^q}^2 \right)^{1/2} \lesssim \|f\|_{L^q}$$

Corollary 2.14.

$$\|f\|_{H^s}^2 \approx \sum 2^{2js}\|\Delta_j f\|_{L^2}^2$$

2.8 Strichartz Estimates for the Homogeneous Wave Equation

Let $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ and suppose

$$\begin{cases} \Box u = 0 \\ u(0, \cdot) = f; \quad \partial_t u(0, \cdot) = g \end{cases}$$

Theorem 2.15. *Let $2 \leq p \leq \infty$, $2 \leq q < \infty$ and*

$$\frac{2}{p} \leq \frac{n-1}{2}\left(1 - \frac{2}{q}\right) \qquad s = \frac{n}{2} - \frac{n}{q} - \frac{1}{p}$$

then

$$\|u\|_{L_t^p L_x^q} \lesssim \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-1}}$$

We will ignore the endpoint case $p = 2, q = -\frac{2(n-1)}{n-3}$.
Note: The restriction $q < \infty$ is not strictly necessary, though we will use it in our proof. The restriction $\frac{2}{p} \leq \frac{n-1}{2}\left(1 - \frac{2}{q}\right)$ is necessary. This is shown by the Knapp counterexample.
To see that the requirement $s = \frac{n}{2} - \frac{n}{q} - \frac{1}{p}$ is necessary, we consider scaling. Recall that

$$\|f(\lambda x)\|_{\dot{H}^s} = \lambda^{s-\frac{n}{2}}\|f\|_{\dot{H}^s}$$

We define $u_\lambda(t,x) = u(\lambda t, \lambda x)$, which gives

$$\Box u_\lambda = 0 \quad u_\lambda(0,x) = f(\lambda x) \quad \partial_t u_\lambda(0,x) = \lambda g(\lambda x)$$

Next we calculate

$$\|u_\lambda\|_{L_t^p L_x^q} = \|u(\lambda t, \lambda x)\|_{L_t^p L_x^q} = \lambda^{-\frac{1}{p}-\frac{n}{q}}\|u(t,x)\|_{L_t^p L_x^q}$$
$$\|f(\lambda x)\|_{\dot{H}^s} = \lambda^{s-\frac{n}{2}}\|f\|\dot{H}^s$$
$$\|\lambda g(\lambda x)\|_{\dot{H}^{s-1}} = \lambda^{s-\frac{n}{2}}\|g\|_{\dot{H}^{s-1}}$$

so that in order for the theorem to have any chance of holding, we need

$$-\frac{1}{p} - \frac{n}{q} = s - \frac{n}{2}$$

2.8.1 Proof Using TT^* Argument

We showed that if u solves the homogeneous wave equation, then

$$\hat{u}(t,\xi) = \cos(t|\xi|)\hat{f}(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\hat{g}(\xi)$$

Note that both terms are linear combinations of $e^{\pm it|\xi|}$. We'll examine

$$e^{\pm it\sqrt{-\Delta}}f = \mathcal{F}^{-1}e^{\pm it|\xi|}\hat{f}(\xi)$$

This solves the half-wave equation

$$\begin{cases} (i\partial_t \pm \sqrt{-\Delta})u = 0 \\ u(0,\cdot) = f \end{cases}$$

Fix $\beta \in C_c^\infty$, supported away from 0, and radial. Set

$$Tf = \int e^{ix\cdot\xi}\beta(\xi)e^{it|\xi|}\hat{f}(\xi)\ d\xi$$

This operator applies $e^{it\sqrt{-\Delta}}$ and localizes at frequency 1.
We'll first prove

$$\|Tf\|_{L_t^p L_x^q} \lesssim \|f\|_{L^2} \tag{4}$$

Let's begin by finding T^*, which is the formal adjoint of T:

$$\langle Tf, F\rangle_{L^2_{tx}} = \langle f, T^*F\rangle_{L^2_x}$$

In other words:

$$\iint Tf \cdot \overline{F}\ dxdt = \int f \cdot \overline{T^*F}\ dx$$

We calculate

$$\begin{aligned}
\iint Tf \cdot \overline{F}\ dxdt &= \iint (Tf)\hat{}\cdot \overline{\hat{F}}\ d\xi dt \quad \text{where the FT is spatial, by Parceval} \\
&= \iint \beta(\xi) e^{it|\xi|} \hat{f}(\xi) \overline{\hat{F}(t,\xi)}\ d\xi dt \\
&= \int \hat{f}(\xi) \overline{\int \overline{\beta(\xi)} e^{-it|\xi|} \hat{F}(t,\xi)\ dt}\ d\xi \\
&= \int f(x) \overline{\iint e^{ix\cdot\xi} \overline{\beta(\xi)} e^{-it|\xi|} \hat{F}(t,\xi)\ dtd\xi}\ dx
\end{aligned}$$

So that we find

$$\begin{aligned}
T^*F &= \iint e^{ix\cdot\xi} \overline{\beta(\xi)} e^{-it|\xi|} \hat{F}(t,\xi)\ d\xi dt \\
&= \int e^{ix\cdot\xi} \overline{\beta(\xi)} \hat{F}(|\xi|,\xi)\ d\xi \quad \text{where the FT is in space-time}
\end{aligned}$$

Recall from Measure theory that if $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then

$$\|f\|_{L^p} = \sup\{|\langle f, g\rangle| : g \in \mathcal{S}', \|g\|_{L^{p'}} \le 1\}$$

So, if $1 \le p, q \le \infty$, then

$$\|F\|_{L^p_t L^q_x} = \sup\{|\langle F, G\rangle : \|G\|_{L^{p'}_t L^{q'}_x} \le 1\}$$

Theorem 2.16. *The following are equivalent:*

1. $T : L^2(\mathbb{R}^n) \to L^p_t L^q_x(\mathbb{R}^{1+n})$ *is bounded.*
2. $T^* : L^{p'}_t L^{q'}_x(\mathbb{R}^{1+n}) \to L^2(\mathbb{R}^n)$ *is bounded.*
3. $TT^* : L^{p'}_t L^{q'}_x(\mathbb{R}^{1+n}) \to L^p_t L^q_x(\mathbb{R}^{1+n})$ *is bounded.*

Proof. (1) $\Rightarrow$ (2)

$$|\langle f, T^*F\rangle| = |\langle Tf, F\rangle| \le \|Tf\|_{L^pL^q}\|F\|_{L^{p'}L^{q'}} \le \|f\|_{L^2}\|F\|_{L^{p'}L^{q'}}$$

Taking the supremum over all f such that $\|f\|_{L^2} \le 1$, we find

$$\|T^*F\|_{L^2} \lesssim \|F\|_{L^{p'}L^{q'}}$$

as desired.

(2) $\Rightarrow$ (1)

$$|\langle Tf, F\rangle| = |\langle f, T^*F\rangle| \le \|f\|_{L^2}\|T^*F\|_{L^2} \lesssim \|f\|_{L^2}\|F\|_{L^{p'}L^{q'}}$$

Taking the supremum over all F such that $\|F\|_{L^{p'}L^{q'}} \le 1$, we find

$$\|Tf\|_{L^pL^q} \lesssim \|f\|_{L^2}$$

as desired.

(1) & (2) $\Rightarrow$ (3)

$$\|T(T^*F)\|_{L^pL^q} \lesssim \|T^*F\|_{L^2} \lesssim \|F\|_{L^{p'}L^{q'}}$$

(3) $\Rightarrow$ (2)

$$\|T^*F\|_{L^2}^2 = \langle T^*F, T*F\rangle = \langle TT^*F, F\rangle \leq \|F\|_{L^{p'}L^{q'}}\|TT^*F\|_{L^pL^q} \lesssim \|F\|_{L^{p'}L^{q'}}^2$$

□

By theorem 2.16, we have that proving (4) is equivalent to showing that

$$\|TT^*F\|_{L^pL^q} \lesssim \|F\|_{L^{p'}L^{q'}}$$

Next we compute $(TT^*F)\check{}(t,\xi)$ and TT^*F:

$$\begin{aligned}
(TT^*F)\check{}(t,\xi) &= e^{it|\xi|}\beta(\xi)(T^*F)\check{} \\
&= e^{it|\xi|}\beta(\xi)\int e^{-is|\xi|}\overline{\beta(\xi)}\hat{F}(s,\xi)ds \\
&= e^{i(t-s)|\xi|}|\beta(\xi)|^2\hat{F}(s,\xi)\ ds \\
TT^*F &= \iint e^{ix\cdot\xi}e^{i(t-s)|\xi|}|\beta(\xi)|^2\hat{F}(s,\xi)\ dsd\xi \\
&= \iiint e^{i(x-y)\cdot\xi}e^{i(t-s)|\xi|}|\beta(\xi)|^2\ d\xi F(s,y)\ dyds \\
&= K*F
\end{aligned}$$

where $K(t-s,x-y) := \int e^{i(x-y)\cdot\xi}e^{i(t-s)|\xi|}|\beta(\xi)|^2\ d\xi$.
Now fix t and look at

$$f \mapsto K_t * f = K(t,\cdot)*f = \iint e^{i(x-y)\cdot\xi}e^{it|\xi|}|\beta(\xi)|^2 f(y)\ d\xi dy$$

Lemma 2.17. *We claim*

(1) $\|K_t*f\|_{L^2} \lesssim \|f\|_{L^2}$

(2) $\|K_t*f\|_{L^\infty} \lesssim \frac{1}{(1+|t|)^{\frac{n-1}{2}}}\|f\|_{L^1}$

Here (1) is essentially an energy estimate and (2) is a dispersive estimate.

Proof. Using the above computations we show

(1)

$$\|K_t*f\|_{L^2} = \|(K_t*f)\check{}\|_{L^2} = \|e^{it|\xi|}|\beta(\xi)|^2\hat{f}\|_{L^2} \lesssim \|\hat{f}\|_{L^2} = \|f\|_{L^2}$$

(2) **Claim**: Take $\hat{d\sigma}(\xi) = \int_{S^{n-1}} e^{i\xi\cdot\omega}\ d\sigma(\omega)$ then $|\hat{d\sigma}(\xi)| \lesssim (1+|\xi|)^{-\frac{n-1}{2}}$.

For now we assume the claim. We have

$$\begin{aligned} K(t,x) &= \int e^{ix\cdot\xi} e^{it|\xi|} |\beta(\xi)|^2\ d\xi \\ &= \int_0^\infty \int_{S^{n-1}} e^{i\rho(x\cdot\omega + t)} a(\rho)\ d\sigma(\omega) d\rho \\ &= \int_0^\infty e^{it\rho} a(\rho) \hat{d\sigma}(\rho x) d\rho \end{aligned}$$

Here $a(\rho) = |\beta(\rho)|^2 \rho^{n-1}$ We consider 2 cases:

Case 1: $|t| > 2|x|$

In this case we have $|x\cdot\omega + t| > |t| - |x| \geq \frac{|t|}{2}$. We note that

$$\int_0^\infty e^{i\rho(x\cdot\omega+t)} a(\rho)\ d\rho = \int_0^\infty \frac{1}{(i(x\cdot\omega+t))^N} \partial_\rho^N (e^{i\rho(x\cdot\omega+t)}) a(\rho)\ d\rho$$

We can use integration by parts to evaluate the integral. Thus

$$\left| \int_0^\infty e^{i\rho(x\cdot\omega+t)} a(\rho)\ d\rho \right| \leq C_N (x\cdot\omega+t)^{-N} \leq C_N 2^N |t|^{-N}$$

Returning to our expression above for $K(t,x)$ we find

$$|K(t,x)| \leq C_N 2^N |t|^{-N} \int_{\mathbb{S}^{n-1}} 1\ d\sigma(\omega) = A_{n-1} C_N 2^N |t|^{-N}$$

where A_{n-1} is the surface area of $\mathcal{S}^{n-1}$.

Case 2: $|t| \leq 2|x|$

Here we evaluate

$$\begin{aligned} |K(t,x)| &\leq \int_0^\infty |\hat{d\sigma}(\rho x)||a(\rho)|\ d\rho \\ &\lesssim \int_0^\infty |\rho x|^{-\frac{n-1}{2}} |a(\rho)|\ d\rho \\ &\leq |x|^{-\frac{n-1}{2}} \int_0^\infty \rho^{-\frac{n-1}{2}} |a(\rho)|\ d\rho \\ &\lesssim (1+|x|)^{-\frac{n-1}{2}} \\ &\leq (1+|t|)^{-\frac{n-1}{2}} \end{aligned}$$

Combining cases 1 and 2 we have

$$\|K_t * f\|_{L^\infty} \leq \|K_t\|_\infty \|f\|_{L^1} \lesssim (1+|t|)^{-\frac{n-1}{2}} \|f\|_{L^1}$$

as desired.

It is left to prove the claim. A full proof is omitted, but we make a a brief argument for $n = 3$. It is left as an exercise for the reader to show $\hat{d\sigma}$ is invariant under rotations. It then suffices to take $\xi = (0, 0, \rho)$ with $\rho = |\xi|$.

$$\begin{aligned}
\hat{d\sigma}(0,0,\rho) &= \int_0^\pi \int_0^{2\pi} e^{i\rho\cos\phi} \sin\phi \; d\theta d\phi \\
&= 2\pi \int_{-1}^1 e^{i\rho u} \; du \\
&= \frac{2\pi}{i\rho} e^{i\rho u}\Big|_{-1}^1 \\
&= \frac{4\pi}{\rho}\left(\frac{e^{i\rho} - e^{-i\rho}}{2i}\right) \\
&= 4\pi \frac{\sin(|\xi|)}{|\xi|}
\end{aligned}$$

We immediately see the desired decay as $|\xi|$ goes to infinity. The Taylor expansion of sin handles the possible singularity at $|\xi| = 0$ so that it remains bounded.

□

Lemma 2.17 along with the Riesz-Thorin interpolation theorem gives for all q, p such that

$$\frac{1}{q} = \frac{s}{2} + \frac{1-s}{\infty} = \frac{s}{2} \qquad \frac{1}{p} = \frac{s}{2} + \frac{1-s}{1} = \frac{1}{q} + 1 - \frac{2}{q} = 1 - \frac{1}{q} = \frac{1}{q'}$$

we have

$$\|K_t * f\|_{L^q} \lesssim \frac{1}{(1+|t|)^{\frac{n-1}{2}(1-s)}}\|f\|_{L^{q'}} = \frac{1}{(1+|t|)^{\frac{n-1}{2}(1-\frac{2}{q})}}\|f\|_{L^{q'}}$$

Returning to our expression for TT^*F we calculate

$$\begin{aligned}
\|TT^*F\|_{L^q} &= \| \int K_{t-s} * F(s,\cdot) \; ds\|_{L^q} \\
&\leq \int \|K_{t-s} * F(s,\cdot) \; ds\|_{L^q} \; ds \\
&\lesssim \int (1+|t-s|)^{-\frac{n-1}{2}(1-\frac{2}{q})} \|F(s,\cdot)\|_{L^{q'}} \; ds \\
\|TT^*F\|_{L^pL^q} &\lesssim \left\| \int (1+|t-s|)^{-\frac{n-1}{2}(1-\frac{2}{q})} \|F(s,\cdot)\|_{L^{q'}} \; ds \right\|_{L^p}
\end{aligned}$$

Case 1: $\frac{2}{p} < \frac{n-1}{2}(1 - \frac{2}{q})$

Since $1 + \frac{1}{p} = \frac{1}{p'} + \frac{2}{p}$, Young's inequality implies

$$\|TT^*F\|_{L^pL^q} \lesssim \left\| \frac{1}{(1+|t|)^{\frac{n-1}{2}(1-\frac{2}{q})}} \right\|_{L^{p/2}} \|F\|_{L^{p'}L^{q'}}$$

Here the $L^{p/2}$ norm on the RHS is finite since $1 < \frac{n-1}{2}(1-\frac{2}{q})\frac{p}{2}$.

Case 2: $\frac{2}{p} = \frac{n-1}{2}(1 - \frac{2}{q})$

Here we have

$$1 + \frac{1}{p} = \frac{1}{p'} + \frac{2}{p} = \frac{1}{p'} + \alpha$$

where $\alpha = \frac{n-1}{2}(1 - \frac{2}{q})$ Thus by the Hardy-Littlewood Fractional Integral Inequality

$$\|TT^*F\|_{L^pL^q} \lesssim \left\| \int \frac{1}{|t-s|^\alpha} \|F(s,\cdot)\|_{L^{q'}} \, ds \right\|_{L^p} \lesssim \|F\|_{L^{p'}L^{q'}}$$

So now we have $\|TT^*F\|_{L^pL^q} \lesssim \|F\|_{L^{p'}L^{q'}}$ which is equivalent to (4) by Theorem 2.16.
Next we let $w(t)f = e^{it\sqrt{-\Delta}}f$, or in other words $\hat{w(t)}f(\xi) = e^{it|\xi|}\hat{f}(\xi)$. Our goal is to show

$$\|w(t)f\|_{L^p_tL^q_x} \lesssim \|f\|_{\dot{H}^s} \tag{5}$$

Fix β as in Littlewood-Paley theory. Then $\sum_{j\in\mathbb{Z}} \beta\left(\frac{\xi}{2^j}\right) = 1$ for all $\xi \neq 0$. We set $\hat{\Delta_j f} = \beta\left(\frac{\xi}{2^j}\right)\hat{f}(\xi)$. Then $f = \sum_{j=-\infty}^{\infty} \Delta_j f$. Furthermore we note that $w(t)$ and Δ_j commute since they are both Fourier multipliers so that

$$w(t)f = \sum w(t)\Delta_j f = \sum \Delta_j w(t) f.$$

We now claim

$$w(t)\Delta_j f = \left[T\left(f\left(\frac{\cdot}{2^j}\right)\right)\right](2^jt, 2^jx)$$

Proof. To prove the claim we calculate

$$\begin{aligned} w(t)\Delta_j f &= \int e^{ix\cdot\xi}e^{it|\xi|}\beta\left(\frac{\xi}{2^j}\right)\hat{f}(\xi)\, d\xi \\ &= 2^{jn}\int e^{i2^jx\cdot\xi}e^{i2^jx\cdot\xi}e^{i2^jt|\xi|}\beta(\xi)\hat{f}(2^j\xi)\, d\xi \\ &= \int e^{i2^jx\cdot\xi}e^{i2^jx\cdot\xi}e^{i2^jt|\xi|}\beta(\xi)\left[f\left(\frac{\cdot}{2^j}\right)\right]\hat{}(2^j\xi)\, d\xi \\ &= \left[T\left(f\left(\frac{\cdot}{2^j}\right)\right)\right](2^jt, 2^jx) \end{aligned}$$

□

Next we find

$$\begin{aligned} \|w(t)\Delta_j f\|_{L^pL^q} &= \left\|T\left(f\left(\frac{\cdot}{2^j}\right)\right)(2^jt, 2^jx)\right\|_{L^pL^q} \\ &= 2^{j(-\frac{1}{p}-\frac{n}{q})}\left\|T\left(f\left(\frac{\cdot}{2^j}\right)\right)(t,x)\right\|_{L^pL^q} \\ &\lesssim 2^{j(-\frac{1}{p}-\frac{n}{q})}\left\|f\left(\frac{\cdot}{2^j}\right)\right\|_{L^2} \\ &= 2^{j(\frac{n}{2}-\frac{1}{p}-\frac{n}{q})}\|f\|_{L^2} \\ &= 2^{js}\|f\|_{L^2} \end{aligned}$$

Note that $\Delta_j \Delta_k = 0$ if $|j-k| > 3$ since Δ_j and Δ_k restrict to different frequencies. Therefore

$$\Delta_j f = \Delta_j \left(\sum_k \Delta_k f \right) = \sum_{|j-k| \le 3} \Delta_j \Delta_k f$$

Thus we have

$$\begin{aligned} \|w(t)\Delta_j f\|_{L^p L^q} &\lesssim \sum_{|j-k|\le 3} \|w(t)\Delta_j \Delta_k f\|_{L^p L^q} \\ &\lesssim \sum_{|j-k|\le 3} 2^{js} \|\Delta_k f\|_{L^2} \\ &\lesssim \sum_{|j-k|\le 3} 2^{ks} \|\Delta_k f\|_{L^2} \end{aligned}$$

This gives

$$\begin{aligned} \|wf\|_{L^p L^q} &\approx \left\| \left(\sum_j (w\Delta_j f)^2 \right)^{1/2} \right\|_{L^p L^q} \\ &\le \left(\sum_j \|w\Delta_j f\|^2_{L^p L^q} \right)^{1/2} \\ &\lesssim \left(\sum_j \sum_{|j-k|\le 3} 2^{2ks} \|\Delta_k f\|^2_{L^2} \right)^{1/2} \\ &\lesssim \left(\sum_k \sum_{|j-k|\le 3} 2^{2ks} \|\Delta_k f\|^2_{L^2} \right)^{1/2} \\ &\approx \|f\|^2_{\dot{H}^s} \end{aligned}$$

This concludes the proof of the Strichartz estimates.

2.9 Strichartz Estimates for the Inhomogeneous Wave Equation

We now turn our attention to the inhomogeneous equation

$$\begin{cases} \Box u = F(t,x) \\ u(0,\cdot) = f(x) \quad \partial_t u(0,\cdot) = g(x) \end{cases}$$

where $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

Theorem 2.18. *Suppose u solves the inhomogeneous wave equation for $n \ge 2$. Furthermore suppose $2 \le p, \tilde{p} \le \infty$ and $2 \le q, \tilde{q} < \infty$ with*

$$\frac{2}{p} \le \frac{n-1}{2}\left(1 - \frac{2}{q}\right) \qquad \frac{2}{\tilde{p}} \le \frac{n-1}{2}$$

$$\frac{n}{q} + \frac{1}{p} = \frac{n}{2} - s = \frac{n}{\tilde{q}'} + \frac{1}{\tilde{p}'} - 2$$

Then

$$\|u\|_{L^p L^q} \lesssim \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-1}} + \|F\|_{\tilde{p}'\tilde{q}'}$$

Additionally we include a simplifying assumption that $(p, q), (\tilde{p}, \tilde{q}) \neq \left(2, \frac{2(n-1)}{n-3}\right)$

Recall that the Fourier transform side of the solution to the inhomogeneous equation is given by

$$\hat{u}(t, \xi) = \hat{f}(\xi) \cos t|\xi| + \hat{g}(\xi) \frac{\sin t|\xi|}{|\xi|} + \int_0^t \frac{\sin(t-s)|\xi|}{|\xi|} \hat{F}(s, \xi) \, ds$$

Using Duhamel's principle and Strichartz estimates for the homogeneous equation, we may assume that $f = g \equiv 0$. Thus, after localizing, it is left to bound.

$$\int_0^t \int e^{ix\cdot\xi} \frac{e^{i(t-s)|\xi|}}{|\xi|} \beta(\xi) \hat{F}(s, \xi) \, d\xi ds \tag{6}$$

We note that

$$TT^* F = \int_{-\infty}^{\infty} \int e^{ix\cdot\xi} e^{i(t-s)|\xi|} |\beta(\xi)|^2 \hat{F}(s, \xi) \, d\xi ds$$

We have already shown $TT^* : L^{p'} L^{q'} \to L^p L^q$ is a bounded linear operator. This is easily extended to $L^{\tilde{p}'} L^{\tilde{q}'} \to L^p L^q$. If we can use this to bound (6), then we are done. This is accomplished by the following lemma.

Lemma 2.19. *Christ-Kiselev Lemma*
Let Y and Z be Banach spaces and assume that $K(t, s)$ is a continuous function that gives a bounded linear map from Y to Z for all t, s. Suppose that $-\infty \leq a < b \leq \infty$, and set

$$Tf(t) = \int_a^b K(t, s) f(s) \, ds$$

Assume that

$$\|Tf\|_{L^q((a,b),Z)} \lesssim \|f\|_{L^p((a,b),Y)}$$

Set

$$Wf(t) = \int_a^t K(t, s) f(s) \, ds$$

Then, if $1 \leq p < q \leq \infty$,

$$\|Wf\|_{L^q((a,b),Z)} \lesssim \|f\|_{L^p((a,b),Y)}$$

Proof. Normalize f so that

$$\|f\|_{L^p((a,b);Y)} = 1$$

WLOG we may assume that $f(s)$ is a continuous function with values in Y and if

$$F(t) = \int_a^t \|f(s)\|_Y^p \, ds$$

then $F:(a,b)\to(0,1)$ is strictly increasing and hence a bijection. It follows that if $I=(c,d)\subset(0,1)$ is an interval, then so is $F^{-1}(I)=(F^{-1}(c),F^{-1}(d))$ and we have

$$\|\chi_{F^{-1}(I)}(s)f(s)\|^p_{L^p((a,b),Y)}=\int_{F^{-1}(c)}^{F^{-1}(d)}\|f(s)\|_Y^p\,ds=F(F^{-1}(d))-F(F^{-1}(c))=|I| \tag{7}$$

Next we consider the set of diadic intervals

$$\{((k-1)2^{-j},k2^{-j});1\le k\le 2^j:j=1,2,3,\dots\}$$

We define a relation between I and J denoted $I\sim J$ if I and J have the same length, I is to the left of J, and I and J are non-adjacent but have adjacent parents. Here we say that $\tilde{I}$ is the parent of I if I is in the set of dyadic intervals for a fixed j, $\tilde{I}$ is in the set of dyadic intervals for $j-1$, and $I\subset\tilde{I}$. Note that if J is fixed, there exists at most 2 I's such that $I\sim J$. Furthermore, for almost every $(x,y)\in(0,1)\times(0,1)$ with $x<y$ there exist unique I and J such that $x\in I$, $y\in J$, and $I\sim J$.
Take $x=F(s)$ and $y=F(t)$. Then for almost every s,t

$$\begin{aligned}\chi_{\{(s,t)\in(a,b)\times(a,b):s<t\}}(s,t)&=\chi_{\{(x,y)\in(0,1)\times(0,1):x<y\}}(x,y)\\&=\sum_{\{I,J:I\sim J\}}\chi_I(x)\chi_J(y)\\&=\sum_{\{I,J:I\sim J\}}\chi_{F^{-1}(I)}(s)\chi_{F^{-1}(J)}(t)\end{aligned}$$

Thus we have

$$Wf=T(\chi_{\{(s,t):s<t\}}f(s))=\sum_{\{I,J:I\sim J\}}\chi_{F^{-1}(J)}(t)T(\chi_{F^{-1}(I)}(s)f(s))$$

Using the Minkowski Integral Inequality we conclude that

$$\|Wf\|_{L^q}\le\sum_{j=2}^{\infty}\|\sum_{\{I,J:I\sim J|I|=2^{-j}\}}\chi_{F^{-1}(J)}T(\chi_{F^{-1}(I)}f)\|_{L^q}$$

Note that

$$\sum_{\{I,J:I\sim J|I|=2^{-j}\}}\chi_{F^{-1}(J)}T(\chi_{F^{-1}(I)}f)=\sum_{\{J:|J|=2^{-j}\}}\sum_{\{I:I\sim J\}}\chi_{F^{-1}(J)}T(\chi_{F^{-1}(I)}f)$$

We use the fact that for each J there are at most 2 I such that $I \sim J$ to find for fixed J

$$\begin{aligned}
\| \sum_{\{I,J: I\sim J |I|=2^{-j}\}} \chi_{F^{-1}(J)} T(\chi_{F^{-1}(I)} f)\|_{L^q} &\leq \| \sum_{\{J:|J|=2^{-j}\}} \sum_{\{I:I\sim J\}} \chi_{F^{-1}(J)} T(\chi_{F^{-1}(I)} f)\|_{L^q} \\
&\leq \| \sum_{\{J:|J|=2^{-j}\}} \chi_{F^{-1}(J)} \sum_{\{I:I\sim J\}} T(\chi_{F^{-1}(I)} f)\|_{L^q} \\
&\leq \|2 \sum_{I:|I|-2^{-j}} T(\chi_{F^{-1}(I)} f)\|_{L^q} \\
&\leq 2 \sum_{I:|I|-2^{-j}} \|T(\chi_{F^{-1}(I)} f)\|_{L^q} \\
&\leq 2 \left(\sum_{I:|I|-2^{-j}} \|T(\chi_{F^{-1}(I)} f)\|_{L^q}^q \right)^{1/q}
\end{aligned}$$

Then by hypothesis and (7) we have

$$\begin{aligned}
\left(\sum_{I:|I|-2^{-j}} \|T(\chi_{F^{-1}(I)} f)\|_{L^q}^q \right)^{1/q} &\lesssim \left(\sum_{I:|I|-2^{-j}} \|\chi_{F^{-1}(I)} f\|_{L^q}^q \right)^{1/q} \\
&= \left(\sum_{I:|I|-2^{-j}} 2^{-jq/p} \right)^{1/q} \\
&= 2^{-j(1/p-1/q)}
\end{aligned}$$

It remains to sum over j, using $p < q$ to show $-(1/p - 1/q) < 0$, to obtain the desired result. □

This concludes our discussion of the linear wave equation. In the following section we turn our attention to the nonlinear equation and discuss the energy critical case. Many of the tools developed in our work with the linear wave equation will be utilized as we explore the nonlinear wave equation

3 Introducing The Nonlinear Wave Equation

The nonlinear wave equation is given by

$$\begin{cases} \Box u = F(u) \\ u(0,\cdot) = f \quad \partial_t u(0,\cdot) = g \end{cases}$$

where $F : \mathbb{R} \to \mathbb{R}$ is a nonlinear function. We will discuss nonlinearities with exponential behavior, paying careful attention to the value of the exponent.

3.1 Small Data Theory

Consider the nonlinear wave equation

$$\begin{cases} \Box u = F_\kappa(u) \\ u(0,\cdot) = f \quad \partial_t u(0,\cdot) = g \end{cases} \tag{8}$$

where

$$F_\kappa \in C^1(\mathbb{R}) \quad |F_\kappa(u)| \lesssim |u|^\kappa \quad |F_\kappa(u)| \lesssim |uF'_\kappa(u)| \lesssim |F_\kappa(u)|$$

In other words, the nonlinearity behaves roughly like $\pm|u|^\kappa$.
We begin by stating some general Sobolev inequalities that will be useful.

Theorem 3.1. *Sobolev Inequalities*

(1) *Let $m \in \mathbb{N}$ and assume $1 \leq p, q < \infty$ satisfy*

$$\frac{1}{p} - \frac{1}{q} = \frac{m}{n}$$

Then we have

$$\|f\|_{L^q(\mathbb{R}^n)} \lesssim \sum_{|\alpha|=m} \|\partial^\alpha f\|_{L^p(\mathbb{R}^n)}$$

(2) *If $p > \frac{n}{m}$ then*

$$\|f\|_{L^\infty(\mathbb{R}^n)} \lesssim \sum_{|\alpha|\leq m} \|\partial^\alpha f\|_{L^p(\mathbb{R}^n)}$$

Example 3.2. *Sobolev Inequalities*

(a) *We showed*

$$\|f\|_{L^\infty(\mathbb{R}^n)} \lesssim \|f\|_{H^s(\mathbb{R}^n)} \qquad \text{if } s > \frac{n}{2}$$

which is equivalent to (2) *when $p = 2$.*

(b) *Consider the case when $n = 3$, $m = 1$, and $p = 2$. Then $q = 6$ and we have*

$$\|f\|_{L^6(\mathbb{R}^3)} \lesssim \|\nabla f\|_{L^2(\mathbb{R}^3)}$$

We check the scaling for this inequality

$$\|f(\lambda x)\|_{L^6(\mathbb{R}^3)} = \lambda^{-\frac{1}{2}}\|f\|_{L^6(\mathbb{R}^3)}$$

$$\|\nabla(f(\lambda x))\|_{L^2(\mathbb{R}^3)} = \lambda\|(\nabla f)(\lambda x)\|_{L^2(\mathbb{R}^3)} = \lambda\lambda^{-\frac{3}{2}}\|\nabla f\|_{L^2(\mathbb{R}^3)} = \lambda^{-\frac{1}{2}}\|\nabla f\|_{L^2(\mathbb{R}^3)}$$

Thus we have the desired scaling.

Theorem 3.3. *Global Existence and Uniqueness for Small Data*
Let $\kappa \geq 3$ and set $\gamma = \frac{3}{2} - \frac{2}{\kappa-1}$. Then there exists $\epsilon > 0$ such that if

$$\|f\|_{\dot{H}^\gamma} + \|g\|_{\dot{H}^{\gamma-1}} < \epsilon$$

then there exists a unique global (weak) solution to (8).

Note that if the size of the data is ϵ, then the linear approximation is roughly epsilon so that $F_\kappa(u) \sim \epsilon^\kappa$. Thus we see heuristically that as κ gets larger, working with this equation should get easier in some sense.
The bound on κ is not sharp for obtaining global solutions to (8). The Strauss conjecture (which has been proved) states that there is a global solution for $\kappa > 1 + \sqrt{2}$. In general we must have $\gamma \geq \frac{3}{2} - \frac{2}{\kappa-1}$ to have a solution. A fact due to John, which we state without proof, gives that for each $\kappa > 1 + \sqrt{2}$ there exists a solution with more restrictions on both the nonlinearity and the data.

Theorem 3.4. *[John]*
For all $\kappa > 1 + \sqrt{2}$ there exists a $g \in C_c^\infty$ such that

$$\begin{cases} \Box u = |u|^\kappa \\ u(0,\cdot) \equiv 0; \partial_t u(0,\cdot) = g \end{cases} \tag{9}$$

has lifespan T_* with $0 < T_* < \infty$.

If u solves (9), then $u_\epsilon = \epsilon^{-\frac{2}{\kappa-1}} u\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right)$ solves the same equation with data $(0, g_\epsilon)$, where $g_\epsilon = \epsilon^{-\frac{2}{\kappa-1}-1} g\left(\frac{x}{\epsilon}\right)$. The lifespan of u_ϵ is then given by ϵT^* and we have

$$\|g_\epsilon\|_{\dot{H}^{\gamma-1}} = \epsilon^{\frac{3}{2}-\gamma-\frac{2}{\kappa-1}} \|g\|_{\dot{H}^{\gamma-1}}$$

Note that the exponent on the RHS is positive if $\gamma < \frac{3}{2} - \frac{2}{\kappa-1}$. Thus as $\epsilon \to 0$, the lifespan, size of the data, and the size of the support shrink.

Exercise 1. *Add translates and dilates of g to show that there does not exist a strip $[0,T] \times \mathbb{R}^3$ where a solution exists if $\gamma < \frac{3}{2} - \frac{2}{\kappa-1}$.*

Recall the inhomogeneous Strichartz estimate

$$\|u\|_{L^p L^q} \lesssim \|u(0,\cdot)\|_{\dot{H}^s} + \|\partial_t u(0,\cdot)\|_{\dot{H}^{s-1}} + \|\Box u\|_{L^{\tilde{p}'} L^{\tilde{q}'}}$$

for the appropriate numerology.

Exercise 2. *Show the stronger bound*

$$\|u\|_{L^p L^q([0,T]\times\mathbb{R}^n)} + \|u(T,\cdot)\|_{\dot{H}^s} + \|\partial_t u(T,\cdot)\|_{\dot{H}^{s-1}} \lesssim \|u(0,\cdot)\|_{\dot{H}^s} + \|\partial_t u(0,\cdot)\|_{\dot{H}^{s-1}} + \|\Box u\|_{L^{\tilde{p}'} L^{\tilde{q}'}([0,T]\times\mathbb{R}^n)}$$

3.1.1 The Range $3 \leq \kappa \leq 5$

Let $3 \leq \kappa \leq 5$ and take $\gamma = \frac{3}{2} - \frac{2}{\kappa-1}$. In this range we have the following Strichartz estimate,

$$\|u\|_{L^{\frac{2\kappa}{(1+\gamma)}} L^{\frac{2\kappa}{(2-\gamma)}}} + \|u(T,\cdot)\|_{\dot{H}^\gamma} + \|\partial_t u(T,\cdot)\|_{\dot{H}^{\gamma-1}} \lesssim \|u(0,\cdot)\|_{\dot{H}^\gamma} + \|\partial_t u(0,\cdot)\|_{\dot{H}^{\gamma-1}} + \|\Box u\|_{L^{\frac{2}{(1+\gamma)}} L^{\frac{2}{(2-\gamma)}}} \tag{10}$$

Proof. (of Theorem 3.3)
We set $u_{-1} \equiv 0$ and let u_j solve

$$\begin{cases} \Box u_j = F_\kappa(u_{j-1}) \\ u_j(0,\cdot) = f; \partial_t u(0,\cdot) = g \end{cases}$$

Step 1: We first use induction to show $\|u_j\|_{L^{\frac{2\kappa}{1+\gamma}}L^{\frac{2\kappa}{2-\gamma}}} \lesssim \epsilon$.

By (10) we have

$$\|u_0\|_{L^{\frac{2\kappa}{1+\gamma}}L^{\frac{2\kappa}{2-\gamma}}} \leq C_0 \left(\|f\|_{\dot{H}^\gamma} + \|g\|_{\dot{H}^{\gamma-1}}\right) < C_0\epsilon$$

For our induction hypothesis, we assume $\|u_{j-1}\|_{L^{\frac{2\kappa}{1+\gamma}}L^{\frac{2\kappa}{2-\gamma}}} \leq 2C_0\epsilon$. Now (10) implies

$$\begin{aligned}
\|u_j\|_{L^{\frac{2\kappa}{1+\gamma}}L^{\frac{2\kappa}{2-\gamma}}} &\leq C_0\epsilon + C\|F_\kappa(u_{j-1})\|_{L^{\frac{2}{1+\gamma}}L^{\frac{2}{2-\gamma}}} \\
&\leq C_0\epsilon + C\||u_{j-1}|^\kappa\|_{L^{\frac{2}{1+\gamma}}L^{\frac{2}{2-\gamma}}} \\
&\leq C_0\epsilon + C\|u_{j-1}\|^\kappa_{L^{\frac{2\kappa}{1+\gamma}}L^{\frac{2\kappa}{2-\gamma}}} \\
&\leq C_0\epsilon + C(2C_0\epsilon)^\kappa \\
&\leq 2C_0\epsilon \qquad \text{provided } C2^\kappa C_0^{\kappa-1}\epsilon^{\kappa-1} \leq 1
\end{aligned}$$

Step 2: Next we show

$$\|u_j - u_{j-1}\|_{L^{\frac{2\kappa}{1+\gamma}}L^{\frac{2\kappa}{2-\gamma}}} \leq \frac{1}{2}\|u_{j-1} - u_{j-2}\|_{L^{\frac{2\kappa}{1+\gamma}}L^{\frac{2\kappa}{2-\gamma}}}$$

This gives us that the sequence is Cauchy. Since L^p spaces are complete, the sequence converges to some $u \in L^{\frac{2\kappa}{1+\gamma}}L^{\frac{2\kappa}{2-\gamma}}$. We note that $(u_{j-1} - u_{j-2})$solves

$$\Box(u_{j-1} - u_{j-2}) = F_\kappa(u_{j-1}) - F_\kappa(u_{j-2})$$

$$(u_{j-1} - u_{j-2})(0, \cdot) \equiv 0 \equiv \partial_t(u_{j-1} - u_{j-2})(0, \cdot)$$

Furthermore

$$\begin{aligned}
F_\kappa(u_{j-1}) - F_\kappa(u_{j-2}) &= \int_0^1 \partial_s \left[F_\kappa(su_{j-1} + (1-s)u_{j-2})\right]\, ds \\
&= \int_0^1 F_\kappa'(su_{j-1} + (1-s)u_{j-2})(u_{j-1} - u_{j-2})\, ds
\end{aligned}$$

By the properties of F_κ' we get

$$\int_0^1 F_\kappa'(su_{j-1} + (1-s)u_{j-2})\, ds \in O(|u_{j-1}|^{\kappa-1} + |u_{j-2}|^{\kappa-1})$$

By (10) again we have

$$\begin{aligned}
\|u_j - u_{j-1}\|_{L^{\frac{2\kappa}{1+\gamma}}L^{\frac{2\kappa}{2-\gamma}}} &\leq C_0\|F_\kappa(u_{j-1}) - F_\kappa(u_{j-2})\|_{L^{\frac{2}{1+\gamma}}L^{\frac{2}{2-\gamma}}} \\
&\leq C\|(|u_{j-1}|^{\kappa-1} + |u_{j-2}|^{\kappa-1})(u_{j-1} - u_{j-2})\|_{L^{\frac{2}{1+\gamma}}L^{\frac{2}{2-\gamma}}} \\
&\leq C\left[\|u_{j-1}\|^{\kappa-1}_{L^{\frac{2\kappa}{1+\gamma}}L^{\frac{2\kappa}{2-\gamma}}} + \|u_{j-2}\|^{\kappa-1}_{L^{\frac{2\kappa}{1+\gamma}}L^{\frac{2\kappa}{2-\gamma}}}\right]\|u_{j-1} - u_{j-2}\|_{L^{\frac{2\kappa}{1+\gamma}}L^{\frac{2\kappa}{2-\gamma}}} \\
&\leq C2(2C_0\epsilon)^{\kappa-1}\|u_{j-1} - u_{j-2}\|_{L^{\frac{2\kappa}{1+\gamma}}L^{\frac{2\kappa}{2-\gamma}}}
\end{aligned}$$

Choosing ϵ sufficiently small so that the coefficient is $\leq \frac{1}{2}$ we complete our argument that u_j is a Cauchy sequence. Note that the limit of this sequence satisfies (8) so that the proof of existence of a solution is completed.

□

3.1.2 The Range $\kappa \geq 5$

Next we consider the range $\kappa \geq 5$. Recall the estimate

$$\|v\|_{L^4L^4} + \|v(T,\cdot)\|_{\dot{H}^{\frac{1}{2}}} + \|\partial_t v(T,\cdot)\|_{\dot{H}^{-\frac{1}{2}}} \lesssim \|v(0,\cdot)\|_{\dot{H}^{\frac{1}{2}}} + \|\partial_t(0,\cdot)\|_{\dot{H}^{-\frac{1}{2}}} + \|\Box v\|_{L^{\frac{4}{3}}L^{\frac{4}{3}}}$$

Apply this to $v = \left(\sqrt{-\Delta}\right)^{\gamma-\frac{1}{2}} u = \left(|\xi|^{\gamma-\frac{1}{2}}\hat{u}(t,\xi)\right)\check{}$.
We claim that if $\Box u = F$ then $\Box v = \left(\sqrt{-\Delta}\right)^{\gamma-\frac{1}{2}} F$

Proof. (of claim) Without loss of generality we take $u(0,\cdot) \equiv 0 \equiv \partial_t u(0,\cdot)$. Then if $\Box u = F$ we have the representation

$$\hat{u}(t,\xi) = \int_0^t \frac{\sin(t-s)|\xi|}{|\xi|}\hat{F}(s,\xi)\, ds$$

multiplying both sides by $|\xi|^{\gamma-\frac{1}{2}}$ gives the desired result. □

We note the equalities

$$\|(\sqrt{-\Delta})^{\gamma-\frac{1}{2}}u(T,\cdot)\|_{\dot{H}^{\frac{1}{2}}} = \|u(T,\cdot)\|_{\dot{H}^{\gamma}} \qquad \|(\sqrt{-\Delta})^{\gamma-\frac{1}{2}}\partial_t u(T,\cdot)\|_{\dot{H}^{-\frac{1}{2}}} = \|\partial_t u(T,\cdot)\|_{\dot{H}^{\gamma-1}}$$

and plugging these into the above bound obtain

$$\|(\sqrt{-\Delta})^{\gamma-\frac{1}{2}}u\|_{L^4L^4}+\|u(T,\cdot)\|_{\dot{H}^{\gamma}}+\|\partial_t u(T,\cdot)\|_{\dot{H}^{\gamma-1}} \lesssim \|u(0,\cdot)\|_{\dot{H}^{\gamma}}+\|\partial_t u(0,\cdot)\|_{\dot{H}^{\gamma-1}}+\|(\sqrt{-\Delta})^{\gamma-\frac{1}{2}}\Box u\|_{L^{\frac{4}{3}}L^{\frac{4}{3}}} \tag{11}$$

In general for $q \geq 4$ we have the following Strichartz estimate:

$$\|v\|_{L^qL^{\frac{3q}{q-1}}} \lesssim \|v(0,\cdot)\|_{\dot{H}^{\frac{1}{2}}} + \|\partial_t v(0,\cdot)\|_{\dot{H}^{-\frac{1}{2}}} + \|\Box v\|_{L^{\frac{4}{3}}L^{\frac{4}{3}}}$$

Then if $\gamma = \frac{3}{2} - \frac{4}{q}$ we have by the Sobolev inequalities

$$\begin{aligned}\|u\|_{L^q_tL^q_x} &\lesssim \||D|^{\gamma-\frac{1}{2}}u\|_{L^qL^{\frac{3q}{q-1}}} \\ &\lesssim \|u(0,\cdot)\|_{\dot{H}^{\gamma}} + \|\partial_t(0,\cdot)\|_{\dot{H}^{\gamma-1}} + \||D|^{\gamma-\frac{1}{2}}\Box u\|_{L^{\frac{4}{3}}L^{\frac{4}{3}}}\end{aligned} \tag{12}$$

where $|D| = \sqrt{-\Delta}$.
In order to prove the existence of solutions we will use the above calculations as well as the following lemma. We save the proof of the lemma for later.

Lemma 3.5. *Let $F \in C^1$ and suppose there exits a C_0 such that*

$$C_0^{-1} \leq \frac{|uF'(u)|}{|F(u)|} \leq C_0$$

Then if $1 < q < r, p < \infty$, $0, \sigma \leq 1$ and $\frac{1}{q} - \frac{1}{p} + \frac{1}{r}$, we have

$$\||D|^{\sigma}F(u)\|_{L^q} \lesssim \|F'(u)\|_{L^p}\||D|^{\sigma}u\|_{L^r}$$

We are now equipped to prove the existence of solutions to (8).

Proof. We set $u_{-1} \equiv 0$ and for $j \geq 0$ define u_j as satisfying

$$\begin{cases} u_j = F_\kappa(u_{j-1}) \\ u_j(0,\cdot) = f \quad \partial_t u_j(0,\cdot) = g \end{cases}$$

Take $\gamma = \frac{3}{2} - \frac{2}{\kappa-1}$ and $q = 2(\kappa - 1)$.

Step 1: Define

$$A_m(T) = \||D|^{1-\frac{2}{\kappa-1}} u_m\|_{L^4L^4(()0,T]\times\mathbb{R}^3)} + \|u_m\|_{L^qL^q((0,T]\times\mathbb{R}^3)}$$

Our goal is to show $A_m(T) \leq 2C_0\epsilon$. By (11) and (12) we have $A_0(T) \leq C_0\epsilon$ and

$$\begin{aligned} A_m(T) &\leq C_0\epsilon + C\||D|^{\gamma-1}F_\kappa(u_{m-1})\|_{L^{\frac{4}{3}}L^{\frac{4}{3}}} \\ &\leq C_0\epsilon + C\|F_\kappa'(u_{m-1})\|_{L^2L^2}\||D|^{\gamma-\frac{1}{2}}u_{m-1}\|_{L^4L^4} \\ &\leq C_0\epsilon + C\|u_{m-1}\|^{\kappa-1}_{L^qL^q}\||D|^{\gamma-\frac{1}{2}}u_{m-1}\|_{L^4L^4} \\ &\leq C_0\epsilon + C(A_{m-1}(T))^\kappa \\ &\leq C_0\epsilon + C(2C_0\epsilon)^\kappa \\ &\leq 2C_0\epsilon \end{aligned}$$

for sufficiently small ϵ.

Step 2: We wish to show $\|u_m - u_{m-1}\|_{L^4L^4} \leq \frac{1}{2}\|u_{m-1} - u_{m-2}\|_{L^4L^4}$ in order to establish $\{u_m\}$ is a Cauchy sequence. Thus the sequence converges, and the limiting function satisfies (8).

$$\begin{aligned} \|u_m - u_{m-1}\|_{L^4L^4} &\leq C_0\|F_\kappa(u_{m-1}) - F_\kappa(u_{m-2})\|_{L^{\frac{4}{3}}L^{\frac{4}{3}}} \\ &\leq C\|(|u_{m-1}|^{\kappa-1} + |u_{m-2}|^{\kappa-1})|u_{m-1} - u_{m-2}|\|_{L^{\frac{4}{3}}L^{\frac{4}{3}}} \\ &\leq C\left(\|u_{m-1}\|_{L^{2(\kappa-1)}_{t,x}} + \|u_{m-2}\|^{\kappa-1}_{L^{2(\kappa-1)}_{t,x}}\right)\|u_{m-1} - u_{m-2}\|_{L^4L^4} \end{aligned}$$

$$\leq C(4C_0\epsilon)^{\kappa-1}\|u_{m-1} - u_{m-2}\|_{L^4L^4}$$

□

A quick note on uniqueness: if u_1 and u_2 are solutions of 8, then an argument similar as that above can be used to show

$$\|u_1 = u_2\|_{L^4L^4} \leq \frac{1}{2}\|u_1 - u_2\|_{L^4L^4}$$

and thus $u_1 = u_2$.

The lemma used in the existence proof is a special case of the following lemma, which we will (eventually) prove in full.

Lemma 3.6. *Let $f : \mathbb{R}^k \to \mathbb{R}^\ell$ be C^1 and assumee $F(0) = 0$. Suppose F satisfies*

$$|F'(\tau v + (1-\tau)w)| \leq \mu(\tau)[G(v) + G(w)]$$

for some $G > 0$ and $\mu \in L^1[0,1]$. Then if $s \in (0,1)$, $q \in (1,\infty)$, and $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$ with $p \in (1,\infty]$ and $r \in (1,\infty)$, we have

$$\||D|^s F(u)\|_{L^q} \lesssim \|F'(u)\|_{L^p}\||D|^s u\|_{L^r}$$

In order to prove this lemma we will need some background work. First, we bring the Hardy-Littlewood Maximal Function to the reader's attention. This function is given by

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{B_r(x)|} \int_{B_r(x)} |f(y)| \, dy$$

We state as fact, if $1 < p \leq \infty$ then

$$\|\mathcal{M}f\|_{L^p} \lesssim \|f\|_{L^p}.$$

Moreover, if $f \in L^1_{loc}$ then

$$\lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y) \, dy = f(x)$$

for almost every x.
We also state as fact the Vector/square function estimate, which is due to Fefferman-Stein: if $1 < p < \infty$, then

$$\| \left(\sum_k (M_{g_k})^2 \right)^{\frac{1}{2}} \|_{L^p} \lesssim \| \left(\sum_k g_k^2 \right)^{\frac{1}{2}} \|_{L^p}$$

Next we recall some notation and a fact from Littlewood-Paley theory:

$$\chi(\xi) = \begin{cases} 1 & |\xi| \leq \frac{1}{2} \\ 0 & |\xi| > 1 \end{cases}$$

$$\beta(\xi) = \chi(\xi) - \chi(2\xi)$$

$$\beta_j(\xi) = \beta(\frac{\xi}{2^j})$$

$$\Delta_j f = (\beta_j(\xi)\hat{f}(\xi))\check{}$$

$$\sum_{j \in \mathbb{Z}} \hat{f}_j(\xi) = \hat{f}(\xi) \quad \forall \xi \neq 0$$

Then if $1 < p < \infty$ we have

$$\|f\|_{L^p} \approx \| \left(\sum_{j \in \mathbb{Z}} (\Delta_j f)^2 \right)^{\frac{1}{2}} \|_{L^p}$$

Lemma 3.7. *Using the definitions of β_j and $\mathcal{M}$ above, we have*

$$\int |v(x) - v(y)||\beta_j(x-y)| \, dy\check{} \leq C(\mathcal{M})v(x) \quad a.e$$

where C is independent of j.

Proof. We consider $\int |v(x)||\beta_j(x-y)| \, dy$ and $\int |v(y)||\beta_j(x-y)| \, dy$ separately.

Step 1 Fix $\epsilon > 0$, then there exists an $r > 0$ such that

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} |v(y)|\ dy \le \epsilon + \mathcal{M}v(x)$$

We also note that

$$|\check{\beta}_j(x)| = 2^{jn}|\check{\beta}(2^j x)| \le \frac{C_N 2^{jn}}{(2+2^j|x|)^N}$$

where the equality follows from a change of variables and the inequality follows from integrating by parts. Integrating both sides (using u substitution with $u = 2^j|x|$ for the RHS), we see clearly that

$$\int |\check{\beta}_j(x)|\ dx \le C$$

where C is independent of j.

Step 2 We now consider

$$\begin{aligned}\int |v(y)||\check{\beta}_j(x-y)|\ dy &\le C_N \int \frac{2^{jn}}{(2+2^j|x-y|)^N}|v(y)|\ dy \\ &= C_N\Bigg[\int_{|x-y|\le 2^{-j}} \frac{2^{jn}}{(2+2^j|x-y|)^N}|v(y)|\ dy \\ &\quad + \sum_{k\ge -j}\int_{2^k\le|x-y|\le 2^{k+1}} \frac{2^{jn}}{(2+2^j|x-y|)^N}|v(y)|\ dy\Bigg]\end{aligned}$$

$$\int_{|x-y|\le 2^{-j}} 2^{jn}|v(y)|\ dy = \frac{1}{2^{-jn}}\int_{|x-y|\le 2^{-j}} |v(y)|\ dy \le (\mathcal{M}v)(x)$$

$$\int_{2^k<|x-y|<2^{k+1}} \frac{2^{jn}}{(2^j|x-y|)^N}|v(y)|\ dy \le \frac{2^{(j+k)n}}{2^{jN}2^{kN}}\frac{1}{2^{kN}}\int_{|x-y|\le 2^{k+1}} |v(y)|\ dy \lesssim C_N 2^{(k+j)(n-N)}(\mathcal{M})v(x)$$

Choosing $N > n$ completes the proof.

□

Lemma 3.8. *The following inequalities hold in general:*

(1) $|\Delta_k v(x) - \Delta_k v(y)| \le C2^k|x-y|\int_0^1 \mathcal{M}v(sx+(1-s)y)\ ds$

(2) $|x-y| \le 2^{-k} \Rightarrow |\Delta_k v(x) - \Delta_k v(y)| \le C2^k|x-y|\mathcal{M}v(x)$

Proof. We first note the following calculation

$$\begin{aligned}|\Delta_k v(x) - \Delta_k v(y)| &\le C\left|\int v(z)\big(\check{\beta}_k(x-z) - \beta_k(y-z)\big)\ dz\right| \\ &= C\left|\int v(z)\int_0^1 (D\check{\beta}_k)(sx+(1-s)y-z)|x-y|\ dsdz\right|\end{aligned}$$

(1) We note
$$|(D\check{\beta}_k)(sx+(1-s)y-z)| = 2^k 2^{kn}|(D\beta)\check{}(2^k(sx+(1-s)y-z))|$$
Then the lemma follows by applying the proof of Lemma 3.7

(2) We have
$$|x-y|\le 2^{-k} \Rightarrow |sx+(1-s)y-z| = |x-z+(1-s)(y-x)| \ge |x-z|-2^{-k}$$
It follows that
$$\frac{1}{\big(2+2^k|sx+(1-s)y-z|\big)^N} \le \frac{1}{\big(1+2^k|x-z|\big)^N}$$
Therefore $|x-y|\le 2^{-k}$ gives
$$\begin{aligned}|\Delta_k v(x)-\Delta_k v(y)| &\le C2^k|x-y|\left|\int v(z)\int_0^1 \frac{2^{kn}}{(1+2^k|x-z|)^N}\, dsdz\right| \\ &= C2^k|x-y|\left|\int v(z)\frac{2^{kn}}{(1+2^k|x-z|)^N}\, dz\right| \\ &\le \mathcal{M}v(x)\end{aligned}$$
where the last inequality follows from the proof of Lemma 3.7

□

Let $\varphi \in C_c^\infty$ be supported away from 0 such that $\varphi(\xi)\equiv 1$ on the support of β. Then set $\varphi_k(\xi)=\varphi\left(\frac{\xi}{2^k}\right)$ and define
$$(\tilde{\Delta}_k u)\hat{} = \varphi_k(\xi)\hat{u}(\xi)$$
Then we have
$$\Delta_k\tilde{\Delta}_k u = \Delta_k u.$$
We can now change part 2 of the previous lemma to
$$|x-y|\le 2^{-k} \Rightarrow |\tilde{\Delta}_k\Delta_k v(x) - \tilde{\Delta}_k\Delta_k v(y)| \le C2^k|x-y|\mathcal{M}(\Delta_k v)$$

Lemma 3.9. *With the above notation we have*
$$\int |\Delta_k v(x)-\Delta_k v(y)||\check{\beta}_j(x-y)|\, dy \le C2^{k-j}\mathcal{M}(\Delta_k v)(x)$$

Proof. Lemma 3.8 part (2) gives
$$\begin{aligned}\int_{|x-y|\le 2^{-k}} |\Delta_k v(x)-\Delta_k v(y)||\check{\beta}_j(x-y)|\, dy &\le C2^k\mathcal{M}(\Delta_k v)(x)\int_{|x-y|\le 2^{-k}} |x-y||\check{\beta}_j(x-y)|\, dy \\ &\lesssim C2^{k-j}\mathcal{M}(\Delta_k v)(x)\int \frac{2^{jn}}{(1+2^j|x-y|)^{N-1}}\, dy \\ &\le C_N\int \frac{|\Delta_k v(x)|+|\Delta_k v(y)|2^{jn}}{(1+2^j|x-y|)^{N-1}2^j|x-y|}\, dy \\ &\le C_N 2^{k-j}\int \frac{2^{jn}}{(1+2^j|x-y|)^N}(\Delta_k v(x)-\Delta_k v(y))\, dy \\ &\lesssim C_N 2^{k-j}\mathcal{M}(\Delta_k v)(x)\end{aligned}$$
where the last inequality follows from Lemma 3.7. □

Lemma 3.10. *Let $H(y) > 0$. Then we have*

$$\int |v(x) - v(y)||\check{\beta}_j(x-y)|H(y)\ dy \lesssim |v(x)|\mathcal{M}H(x) + \mathcal{M}(vH)(x)$$

Proof. Omitted. Same as proof of Lemma 3.7 □

Lemma 3.11. *Let $H(y) > 0$. Then*

$$\int |\Delta_k v(x) - \Delta_k v(y)||\check{\beta}_j(x-y)|H(y)\ dy \lesssim 2^{k-j}\mathcal{M}(\Delta_k v)(x)\mathcal{M}H(x) + 2^{k-j}\mathcal{M}(|\Delta_k v|H)(x)$$

Proof.

$$\int_{|x-y|\le 2^{-k}} |\Delta_k v(x) - \Delta_k v(y)|H(y)|\check{\beta}_j(x-y)|\ dy \lesssim 2^k \mathcal{M}v(x)\int |x-y|H(y)|\check{\beta}_j(x-y)|\ dy \lesssim 2^{k-j}\mathcal{M}v(x)\mathcal{M}H(x)$$

$$\begin{aligned}\int_{|x-y|\ge 2^{-k}} |\Delta_k v(x) - \Delta_k v(y)|H(y)|\check{\beta}_j(x-y)|\ dy &\lesssim |\Delta_k v(x)|\int \frac{H(y)2^{jn}}{(1+2^j|x-y|)^N}\ dy + \int \frac{|\Delta_k v(y)|H(y)2^{jn}}{(1+2^j|x-y|)^N}\ dy\\ &\lesssim 2^{k-j}\left(\Delta_k v(x)\mathcal{M}H(x) + \mathcal{M}(|\Delta_k v|H)(x)\right)\end{aligned}$$

□

We now return to the proof of Lemma 3.6.

Proof. We begin by noting

$$\int \check{\beta}_j(y)\ dy = \int e^{-0\cdot y}\check{\beta}_j(y)\ dy = \beta_j(0) = 0$$

Thus we have for $H(z) = G\circ u(z)$:

$$\begin{aligned}|\Delta_j(F\circ u)(x)| &\left|\int \check{\beta}_j(x-y)(F\circ u)(y)\ dy\right|\\ &= \left|\int \check{\beta}_j(x-y)\big(F\circ u)(x) - (F\circ u)(y)\big)\ dy\right|\\ &\lesssim \int |\check{\beta}_j(x-y)|(H(x)+H(y))|u(x)-u(y)|\ dy\\ &= H(x)\int |\check{\beta}_j(x-y)||u(x)-u(y)|\ dy + \int |\check{\beta}_j(x-y)|H(y)|u(x)-u(y)|\ dy\\ &=: I + II\end{aligned}$$

$$\begin{aligned}I &\lesssim \sum_{k\ge j} H(x)\int |\check{\beta}_j(x-y)||\Delta_k u(x) - \Delta_k u(y)|\ dy\\ &\quad + \sum_{k<j} H(x)\int |\check{\beta}_j(x-y)||\Delta_k u(x) - \Delta_k u(y)|\ dy\\ &\lesssim \sum_{k\ge j} H(x)\mathcal{M}(\Delta_k u)(x) + \sum_{k<j} 2^{k-j}H(x)\mathcal{M}(\Delta_k v)(x)\end{aligned}$$

where the last inequality follows from Lemma 3.7 and Lemma 3.9.

$$II \lesssim \sum_{k\geq j} \int |\check{\beta}_j(x-y)| H(y) |\Delta_k u(x) - \Delta_k u(y)| \, dy + \sum_{k<j} \int |\check{\beta}_j(x-y)| H(y) |\Delta_k u(x) - \Delta_k u(y)| \, dy$$
$$\lesssim \sum_{k\geq j} \left(|\Delta_k u(x)| \mathcal{M}H(x) + \mathcal{M}(|\Delta_k u| H)(x) \right) + \sum_{k<j} 2^{k-j} \left[\mathcal{M}(\Delta_k u)(x) \mathcal{M}H(x) + \mathcal{M}(|\Delta_k u| H)(x) \right]$$

where the last inequality follows from Lemma 3.10 and Lemma 3.11. Next we calculate

$$\sum_{j\in\mathbb{Z}} 2^{2js} |\Delta_j F \circ u(x)|^2 \lesssim (\mathcal{M}H(x))^2 \left(\sum_j 2^{2js} \left(\sum_{k\geq j} \mathcal{M}(\Delta_k u)(x) \right)^2 + \sum_j 2^{2js} \left(\sum_{k<j} 2^{k-j} \mathcal{M}(\Delta_k u)(x) \right)^2 \right)$$
$$+ \sum_j 2^{2js} \left(\sum_{k\geq j} \mathcal{M}(|\Delta_k u| H)(x) \right)^2 + \sum_j 2^{2js} \left(\sum_{k<j} 2^{k-j} \mathcal{M}(|\Delta_k u| H)(x) \right)^2$$

We note

$$\sum_j 2^{2js} \left(\sum_{k\geq j} \mathcal{M}(\Delta_k u)(x) \right)^2 = \sum_j \left(\sum_{k\geq j} 2^{s(j-k)} s^{sk} \mathcal{M}(\Delta_k u)(x) \right)^2$$
$$= \sum_j \left(\sum_k 2^{-s|j-k|} (2^{ks} \mathcal{M}(\Delta_k u)(x)) \right)^2$$
$$\lesssim \sum_k 2^{2ks} \left(\mathcal{M}(\Delta_k u)(x) \right)^2$$

where the last inequality follows from the assumption $s > 0$ and Young's inequality.

$$\sum_j 2^{2js} \left(\sum_{k<j} 2^{k-j} \mathcal{M}(\Delta_k u)(x) \right)^2 = \sum_j \left(\sum_{k<j} 2^{-(1-s)|k-j|} 2^{ks} \mathcal{M}(\Delta_k u)(x) \right)^2$$
$$\lesssim \sum_k 2^{2ks} \left(\mathcal{M}(\Delta_k u)(x) \right)^2$$

Then we have

$$\| |D|^s (F \circ u)(x) \|_{L^q} \lesssim \left\| \left[\sum_j 2^{2js} (\Delta_j F \circ u(x))^2 \right]^{\frac{1}{2}} \right\|_{L^q}$$
$$\lesssim \left\| \mathcal{M}H(x) \left(\sum_k 2^{2ks} (\mathcal{M}(\Delta_k u)(x))^2 \right)^{\frac{1}{2}} \right\|_{L^q} + \left\| \left(\sum_k 2^{ks} (\mathcal{M}(|\Delta_k u| H)(x))^2 \right)^{\frac{1}{2}} \right\|_{L^q}$$

where the first inequality comes from Littlewood-Paley theory.

$$\left\| \mathcal{M}H(x)\left(\sum_k 2^{2ks}(\mathcal{M}(\Delta_k u)(x))^2\right)^{\frac{1}{2}}\right\|_{L^q} \lesssim \|\mathcal{M}H(x)\|_{L^p}\left\|\left(\sum_k 2^{2ks}(\mathcal{M}(\Delta_k u)(x))^2\right)^{\frac{1}{2}}\right\|_{L^r}$$

$$\lesssim \|H\|_{L^p}\left\|\left(\sum_k 2^{2ks}(\Delta_k u)^2(x)\right)^{\frac{1}{2}}\right\|_{L^r} \lesssim \|H\|_{L^p}\||D|^s u\|_{L^r}$$

where the last inequality comes from Littlewood-Paley theory.

$$\left\|\left(\sum_k 2^{ks}(\mathcal{M}(|\Delta_k u|H)(x))^2\right)^{\frac{1}{2}}\right\|_{L^q} \lesssim \left\|\left(\sum_k 2^{2ks}|\Delta_k u(x)|^2\right)^{\frac{1}{2}} H(x)\right\|_{L^q}$$

$$\lesssim \|H\|_{L^p}\left\|\left(\sum_k 2^{2ks}(\Delta_k u)^2(x)\right)^{\frac{1}{2}}\right\|_{L^r}$$

$$\lesssim \|H\|_{L^p}\||D|^s u\|_{L^r}$$

where the last inequality follows from Littlewood-Paley theory. □

This ties up the loose ends in the proof of the existence of solutions for the range $\kappa > 5$.

3.2 An overview of the Defocusing and Focusing Wave Equations

Before exploring the energy critical defocusing wave equation we make some notes on the contrast with the energy critical focusing wave equation. The focusing wave equation is as above except with the sign change: $(\partial_t^2 - \Delta)u = u^5$. We state some facts about the focusing equation:

(1) There can be blow-up for C_c^∞ data.

For example, take

$$u = \left(\frac{3}{4}\right)^{\frac{1}{4}}(1-t)^{\frac{1}{2}}$$

Then $u_{tt} = u^5$ and $\Delta u = 0$. To get blow up for C_c^∞ data take the backward light cone with point at $t = 1, x = 0$ with initial data that is supported there then we get blow up as t approaches 1 for $\|\partial_t u\|_{L^2}$

(2) Stationary Solutions

There are nonzero stationary solutions $-\Delta h = h^5$. But in the defocusing case, these cannot exist. If a solution to $\Delta h = h^5$ did exist we would have

$$\int |\nabla h|^2 \, dx = \int -(\Delta h)h \, dx = \int -h^5 \cdot h \, dx = -\int h^6 \, dx$$

which implies $h = 0$.

(3) Conserved Energy

We have $\Box u = \mp u^5$ gives

$$E(u)(t) = \int \frac{1}{2}|u'(t,x)|^2 \pm \frac{1}{6}u^6 \, dx$$

is conserved. For the defocusing case this is a coercive energy that is strictly nonnegative.

3.3 General Data for the Defocusing Wave Equation

The defocusing energy critical wave equation is given by

$$\begin{cases} (\partial_t^2 - \Delta)u = -u^5 \\ u(0,\cdot) = f; \quad \partial_t u(0,\cdot) = g \end{cases} \tag{13}$$

Here we consider $(t,x) \in \mathbb{R} \times \mathbb{R}^3$.

3.3.1 Energy Critical and Subcritical Wave Equations

Suppose u solves (13). Let $u_\lambda(tx) = \lambda^{\frac{1}{2}} u(\lambda t, \lambda x)$. Then we have

$$\Box u_\lambda = \lambda^{\frac{1}{2}} \lambda^2 (\Box u)(\lambda t, \lambda x) = -\lambda^{\frac{5}{2}} (u(\lambda t, \lambda x))^5 = -u_\lambda^5$$

Furthermore we have $u_\lambda(0,x) = \lambda^{\frac{1}{2}} f(\lambda x)$. We note

$$\lambda^{\frac{1}{2}} \|\nabla(f(\lambda x))\|_{L^2} = \lambda^{\frac{3}{2}} \|(\nabla f)(\lambda x)\|_{L^2} = \|\nabla f\|_{L^2}$$

Similarly

$$\|\partial_u u_\lambda(0,\cdot)\|_{L^2} = \|\partial_t u(0,\cdot)\|_{L^2}$$

Thus we see that the energy is preserved under scaling.
We state the following standard local existenc theorem.

Theorem 3.12. *Suppose u satisfies*

$$\begin{cases} \Box u = F(u(t,x)) \\ u(0,x) = f(x); \quad \partial_t u(0,x) = g(x) \end{cases} \tag{14}$$

and assume $F \in C^2, F(0) = 0$ and $f \in C_0^3, g \in C_0^2$.
Then there exists $T > 0$ such that we have $u \in C^2([0,T] \times \mathbb{R}^3)$ solving (14). Moreover, if T_* is the sup of all such T and $T_* < \infty$ then $\|u(t,\cdot)\|_{L^\infty} \to \infty$ as $t \nearrow T_*$.

We now look at

$$\begin{cases} \Box u = -u^\kappa \\ u(0,\cdot) = f; \quad \partial_t u(0,\cdot) = g \end{cases} \tag{15}$$

for $1 < \kappa \le 5$.

Proposition 3.13. *In this case we have the same result as above as well as $T^* = \infty$ or $u \notin L^4_t L^{12}_x([0, T_*) \times \mathbb{R}^3)$*

Proof. Suppose u is a solution to(15) on $[0, T_*) \times \mathbb{R}^3$ such that $u \in L^4_t L^{12}_x$, then we need to show $u \in L^\infty_{t,x}([0, T_*) \times \mathbb{R}^3)$.
Fix R such that $u(0, x), \partial_t u(0, x) = 0$ if $|x| > R$. Then $u(t, x) = 0$ if $|x| > t + R$. Take $0 \le t_0 < s < T_*$. We have

$$\sup_{t_0 \le t < s} \|\partial_x^{\le 1} u(t, \cdot)\|_{L^6} \lesssim \sup_{[t_0,s)} \|\partial_x^{\le 1} \partial_x u(t, \cdot)\|_{L^2} \quad \text{by Sobolev embeddings}$$

$$\& \lesssim \|\partial_x^{\le 1} u'(t_0, \cdot)\|_{L^2} + \||partial_x^{\le 1} \Box u\|_{L^1 L^2}$$

$$\le C(t_0, R) + C\||u|^{\kappa-1} \partial_x^{\le 1} u\|_{L^1 L^2}$$

Case 1: Energy Critical ($\kappa = 5$)

$$\sup_{[t_0,s)} \|\partial_x^{\le 1} u(t, \cdot)\|_{L^6} \le C(t_0, R) + C\||u|^4\|_{L^1 L^3} \sup_{[t_0,s)} \|\partial^{\le 1} u(t, \cdot)\|_{L^6}$$

$$= C(t_0, R) + C\|u\|^4_{L^4 L^{12}} \sup_{[t_0,s)} \|\partial^{\le 1} u(t, \cdot)\|_{L^6}\|^6_L$$

Since $\|u\|_{L^4 L^{12}}$ is finite by hypothesis, we have $\|u\|_{L^4 L^{12}([t_0,s) \times \mathbb{R}^3} < \epsilon$ if $T_* - t_0 << 1$. If $T_* - t_0 << 1$, then

$$\sup_{[t_0,s)} \|\partial^{\le 1} u(t, \cdot)\|_{L^6} \le C(t_0, R) + \frac{1}{2} \sup_{[t_0,s)}$$

which gives

$$\sup_{[t_0,s)} \|\partial^{\le 1} u(t, \cdot)\|_{L^6} \le 2C(t_0, R)$$

Finally by Sobolev embeddings and letting s increase to T_* we have

$$\sup_{[t_0,T_*]} \|u(t, \cdot)\|_{L^\infty}$$

as desired. The interval $[0, t_0]$ is trivial.
Case 2: Energy subcritial ($1 < \kappa < 5$)

$$\sup_{[t_0,s)} \|\partial^{\le 1} u(t, \cdot)\|_{L^6} \le C(t_0, R) + C\||u|^{\kappa-1}\|_{L^1 L^3} \sup_{[t_0,s)} \|\partial^{\le 1} u(t, \cdot)\|_{L^6}$$

$$= C(t_0, R) + C\|u\|^{\kappa-1}_{L^{\kappa-1} L^{3(\kappa-1)}} \sup_{[t_0,s]} \|\partial^{\le 1} u\|_{L^6}$$

Since $\kappa - 1 < 4$ and thus $3(\kappa - 1) < 12$

$$\|u\|^{\kappa-1}_{L^{\kappa-1} L^{3(\kappa-1)}} \le C_{T_*,R} \|u\|_{L^4 L^12}$$

The desired result then follows as in the proof of case 1. $\square$

3.3.2 Subcritical Global Existence

We consider the equation

$$\begin{cases} \Box u = -u^\kappa \\ u(0,\cdot) = f; \quad \partial_t u(0,\cdot) = g \end{cases} \tag{16}$$

with $1 < \kappa < 5$.

Proposition 3.14. *If $u \in C^2([0,T_*) \times \mathbb{R}^3)$ solves (16) for $0 < T_* < \infty$ and $g(x) = 0 = g(x)$ if $|x| > R$. Then*

$$E(u,t) = \int \frac{1}{2}|u'(t,x)|^2 + \frac{1}{\kappa+1}u^{\kappa+1}(t,x)\ dx = E(u,0)$$

Proof.

$$\begin{aligned} \frac{d}{dt}E(u,t) &= \int u_t u_{tt} + \sum_{j=1}^3 u_j u_{tj} + u^\kappa u_t\ dx \\ &= \int u_t(u_{tt} - \Delta u + u^\kappa)\ dx \\ &= 0 \end{aligned}$$

□

Lemma 3.15. *Let $0 < C_0 < \infty$ and assume y satisfies $0 \le y(s) \in C^([a,b)$, $y(a) = 0$, and for some $\sigma > 0$*

$$y(x) \le C_0 + \epsilon(y(s))^\sigma$$

Then if $\epsilon < 2^{-\sigma}C_0^{1-\sigma}$, then

$$y(s) \le 2C_0 \quad \forall s \in [a,b)$$

Proof. Define $A = \{s \in [a,b) : y(s) \le 2C_0\}$ We wish to show that A is nonempty, closed and open. This will give $A = [a,b)$ as desired.
nonempty: We have

$$y(a) = 0 \le 2C_0 \Rightarrow A \ne \emptyset$$

closed: Note that

$$A = y^{-1}((-\infty, 2C_0])$$

Since y is continuous, A is closed.
open: Fix $\delta > 0$ such that $(1+\delta)^\sigma = \epsilon^{-1}2^{-\sigma}C_0^{1-\sigma}$. Fix $s \in A$. Since y is continuous there exits $\tilde{\delta} > 0$ such that

$$|s-t| < \tilde{\delta}, t \in [a,b) \Rightarrow y(t) \le y(s) + 2C_0\delta \le 2C_0(1+\delta)$$

But

$$y(t) \le C_0 + \epsilon(y(t))^\sigma \le C_0 + \epsilon(2C_0(1+\delta))^\sigma \le 2C_0$$

Thus A is open. □

Theorem 3.16. ***Global Existence, Subcritical Case***
Let $1<\kappa<5$ and consider the equation

$$\begin{cases}\Box u=-|u|^\kappa \\ u(0,\cdot)=f; \quad \partial_t u(0,\cdot)=g\end{cases} \tag{17}$$

with $f\in C^3, g\in C^2$, and $(t,x)\in\mathbb{R}_+\times\mathbb{R}^3$. Then there exists a global solution u to (17) with $u\in C^2(\mathbb{R}_+\times\mathbb{R}^3)$.

Proof. Recall

$$E(u,t)=\int\frac{1}{2}|u'(t,x)|^2+\frac{1}{\kappa+1}|u|^{\kappa+1}(t,x)\,dx=E(u,0)$$

WLOG we can take the data to be compactly suported. To see this, fix $\chi\in C_c^\infty$ such that $\chi\equiv 1$ on $|x|\le 1$. Set $f_R=\chi\left(\frac{x}{R}\right)f$ and $g_R=\chi\left(\frac{x}{R}\right)g$. Assume that the teorem is true for compactly supported data. Then there exist global solutions u_R to (17) with initial data f_R, g_R. We claim $u_R\to u$ as $R\to\infty$.
To justify the claim, fix $t_0\in\mathbb{R}$ and define

$$\Lambda_{t_0}=\{(t,x):0\le t\le t_0;|x|\le t_0-t\}$$

Then $u_{R_1}=u_{R_2}$ in Λ_{t_0} if $R_1,R_2>t_0$ since $f_{R_1}=f_{R_2}$ and $g_{R_1}=g_{R_2}$ in $|x|\le t_0$. Note that $\mathbb{R}^{1+3}=\cup\Lambda_{t_0}$ so that u_R must converge to u.
Thus WLOG assume the data $\equiv 0$ if $|x|>R$. Fix $0<T_*<\infty$, and take the local solution $u\in C^2([0,T_*]\times\mathbb{R}^3)$. Then we need to show $u\in L_t^4L_x^12([0,T_*]\times\mathbb{R}^3)$. Take $0<t_0<s<T_*$. Then Strichartz estimates give

$$\begin{aligned}\|u\|_{L_t^4L_x^{12}([0,T_*)\times\mathbb{R}^3)}&\le C\|u'(t_0,\cdot)\|_{L^2}+C\||u|^\kappa\|_{L_t^1L_x^2([t_0,s]\times\mathbb{R}^3)}\\&\le C(E(u,0))^{\frac12}+C\||u|^{\kappa-1}\|_{L_t^{\frac{4}{\kappa-1}}L_x^{\frac{12}{\kappa-1}}}\|u\|_{L_t^{\frac{4}{5-\kappa}}L_x^{\frac{12}{7-\kappa}}}\\&=C(E(u,0))^{\frac12}+C\|u\|^{\kappa-1}_{L_t^4L_x^{12}}\|u\|_{L_t^{\frac{4}{5-\kappa}}L_x^{\frac{12}{7-\kappa}}}\end{aligned}$$

ASIDE:

$$\mu(\mathrm{supp}f)<\infty;p<q,\frac{1}{p}=\frac{1}{q}+\frac{1}{a}\Rightarrow\|f\|_{L^p}<[\mu(\mathrm{supp}f)]^{\frac{1}{a}}\|f\|_{L^q}$$

$$\begin{aligned}\|u\|_{L_t^{\frac{4}{5-\kappa}}L_x^{\frac{12}{7-\kappa}}}&\le(T_*-t_0)^{\frac{5-\kappa}{4}}\sup_{[t_0,s]}\|u(t,\cdot)\|_{L^{\frac{12}{7-\kappa}}(\{|x|\le T_*+R\})}\\&\le(T_*-t_0)^{\frac{5-\kappa}{4}}(T_*+R)^{\frac{7-\kappa}{12}-\frac{1}{\kappa+1}}\sup\|u(t,\cdot)\|_{L^{\kappa+1}}\\ \|u\|_{L_t^4L_x^{12}}&\le C(E(u,o))^{\frac12}+C_{T_*,R}(T_*-t_0)^{\frac{5-\kappa}{4}}\|u\|^{\kappa-1}_{L^4L^{12}}\sup\|u(t,\cdot)\|_{L^{\kappa+1}}\\&\le CE^{\frac12}+C_2E^{\frac{1}{\kappa+1}}(T_*-t_0)^{\frac{5-\kappa}{4}}\|u\|^{\kappa-1}_{L_t^4L_x^{12}}\end{aligned}$$

Let $t_0\nearrow T_*$. Then by Lemma 3.15 we have

$$\|u\|_{L^4L^{12}([t_0,s])}\le 2CE^{\frac12}$$

Now let $s\nearrow T_*$. Then we have

$$\|u\|_{L^4L^{12}([t_0,T_*])}\le 2CE^{\frac12}$$

Local existence implies $u\in L^4L^{12}([0,t_0])$. Thus, $u\in L^4L^{12}([0,T_*))$. Since T_* was arbitrary, this implies that u is a global solution. □

3.3.3 The Energy Critical Case

In the proof of global existence for the energy subcritical case it was crucial that $\kappa < 5$. If $\kappa = 5$ then we would fail to get ϵ as $t_0 \nearrow T_*$ in the above proof.
Note that as before we may assume $f(x) = 0 = g(x)$ if $|x| > R$. We consider the backward lightcone with point $T^* + \delta$ and define the following

$$\Lambda(\delta; t_0, s) = \{(t,x) : t_0 \le t \le s; |x - x_0| \le \delta + T_* - t\}$$
$$D_t = \{x : |x - x_0| \le \delta + T_* - t\} \quad t \in [t_0, s]$$
$$\partial\Lambda = M_{t_0}^s = \{(t,x) : t_0 \le t \le s; |x - x_0| = \delta + T_* - t\}$$
$$E(u, D_t) := \int_{D_t} \frac{1}{2}|u'(t,x)| + \frac{1}{6}|u|^6 \, dx$$

Claim: $E(u, D_{t_0}) = E(u, D_s) + \text{ Flux}(u, M_{t_0}^s)$
To prove the claim, set

$$e(u) = \left(\frac{1}{2}|u'|^2 + \frac{1}{6}u^6, -\partial_t u \nabla u\right) \in \mathbb{R}^{1+3}$$

We calculate

$$\begin{aligned}
\text{div}_{t,x} e(u) &= \partial_t\left(\frac{1}{2}|u'|^2 + \frac{1}{6}u^6\right) + \sum_{j=1}^{3} \partial_j(-\partial_t u \partial_j u) \\
&= u_t u_{tt} + \sum_{j=1}^{3} u_j u_{jt} + u^5 u_t - \sum_{j=1}^{3} u_j u_{jt} - \sum_{j=1}^{3} u_t u_{jj} \\
&= u_t[u_{tt} - \Delta u + u^5] = 0
\end{aligned}$$

Thus we find

$$\begin{aligned}
0 &= \int_{\Lambda(\delta,t_0,s)} \text{div} e(u) \\
&= \int_{D_s} \frac{1}{2}|u'(s,x)|^2 + \frac{1}{6}(u(s,x))^6 \, dx - \int_{D_{t_0}} \frac{1}{2}|u'(t_0,x)|^2 + \frac{1}{6}(u(t_0,x))^6 \, dx + \int_{M_{t_0}^s} e(u) \cdot \vec{\nabla} \, d\sigma \\
&= E(u, D_s) - E(u, D_{t_0}) + \text{ Flux}
\end{aligned}$$

Note that

$$\begin{aligned}
\text{Flux } &= \frac{1}{\sqrt{2}} \int_{M_{t_0}^s} \frac{1}{2}|u'(t,x)|^2 + \frac{1}{6}(u(t,x))^6 - \partial_t u \frac{x}{|x|} \cdot \nabla u \, dx \\
&= \frac{1}{\sqrt{2}} \int_{M_{t_0}^s} \frac{1}{2}\left|\frac{x}{|x|}\partial_t u - \nabla u\right|^2 + \frac{1}{6}(u(t,x))^6 \, dx \\
&\ge 0
\end{aligned}$$

Thus we have $E(u, D_s) \le E(u, D_{t_0})$ so that the map $t \to E(u, D_t)$ is non-increasing in $[0, T_*)]$ and $E(u, D_t) \ge 0$. It follows that $E(u, D_t)$ has a limit as $t_0 \to T_*$. Sending $t_0 \to T_*$ and $s \to T_*$ then we have $\text{Flux}(u, M_t^{T_*}) \to 0$ as $t \to T_*$.

Proposition 3.17. *Suppose*

$$\int_{|x-x_0|\le T_*-t_0} \frac{1}{2}|u'(t,x)|^2 + \frac{1}{6}u^6(t,x)\ dx < \epsilon$$

Then if $\epsilon > 0$ small, $u \in L^4_t L^{12}_x(\Lambda(\delta, t_0, T_*))$. If $\delta > 0$ small, $T_* - t_0$ is small.

Proof. The note on δ is automatic from the setup. If $\delta > 0$ is sufficientlly small then

$$\int_{|x-x_0|\le T_*-t_0+\delta} \frac{1}{2}|u'(t_0,x)|^2 + \frac{1}{6}u^6(t_0,x)\ dx < 2\epsilon$$

Then Strichartz estimates give

$$\begin{aligned}
\|u\|_{L^4L^{12}(\Lambda)} &\le C\|u'(t_0,\cdot)\|_{L^2_x} + C\|u^5\|_{L^1L^2(\Lambda)} \\
&\le C(2E)^{\frac{1}{2}} + C\left(\sup_{[t_0,T_*]}\|u(t,\cdot)\|_{L^6_X}\right)\|u^4\|_{L^1_xL^3_x(\Lambda)} \\
&\le C(2E)^{\frac{1}{2}} + C(tE(u,D_{t_0}))^{\frac{1}{6}}\|u\|^4_{L^4_tL^{12}_x(\Lambda)} \\
&\le C(2E)^{\frac{1}{2}} + \epsilon\|u\|^4_{L^4_tL^{12}_x(\Lambda)}
\end{aligned}$$

By Lemma 3.15 this gives

$$\|u\|_{L^4L^{12}(\Lambda)} \le 2C(2E)^{\frac{1}{2}}$$

if ϵ small enough, as desired. □

If we could show

$$\lim_{t\nearrow T_*}\int_{D_t} \frac{1}{2}|u'(t,x)|^2 + \frac{1}{6}u^6(t,x)\ dx = 0$$

then we'll be done.

Proposition 3.18.

$$\lim_{t\nearrow T_*}\int_{|x-x_0|<T_*-t} \frac{1}{6}u^6\ dx = 0 \Rightarrow \lim_{t\nearrow T_*}\int_{|x-x_0|<T_*-t_0} \frac{1}{2}|u'(t,x)|^2\ dx = 0$$

Proof. The previous proof only used smallness from the L^6 portion. The hypothesis thus gives $u \in L^4_tL^{12}_x(\Lambda(0,0,T_*))$. We note

$$\Box u = -u^5 \Rightarrow \Box u' = -5u^4u'$$

and recall

$$\|f\|_{L^6(\mathbb{R}^3)} \lesssim \|\nabla f\|_{L^2(\mathbb{R}^3)}.$$

Now we calculate

$$\begin{aligned}
\sup_{t_0\le t\le s}\{\|u'(t,\cdot)\|_{L^6(D_t)}\} &\le C\sum_{\|\alpha|\le 1}|\partial^\alpha u'(t_0,\cdot)\|_{L^2} + C\|u^4\cdot u'\|_{L^1L^2(\Lambda(0;t_0,s))} \\
&\le C(t_0) + C\left(\sup_{[t_0,s]}\|u'(t,\cdot)\|_{L^6(D_t)}\right)\|u^4\|_{L^1L^3(\Lambda(0;t_0,s))} \\
&= C(t_0) + C\|u\|^4_{L^4L^{12}(\Lambda)}\sup_{[t_0,s]}\|u'(t,\cdot)\|_{L^6(D_t)}
\end{aligned}$$

Since $u \in L^4_t L^{12}_x$ we have that $T_* - t << 1 \Rightarrow \|u\|^4_{L^4 L^{12}(\Lambda)} < \frac{1}{2}$. thus we find

$$\sup_{t_0 \leq t \leq s} \|u'(t,\cdot)\|_{L^6} \leq 2C(t_0)$$

We find

$$\|u'(t,\cdot)\|_{L62(D_t)} \leq (Vol(D_t))^{\frac{1}{3}} \|u'(t,\cdot)\|_{L^6(D_t)} \approx (T_* - t)\|u'(t,\cdot)\|_{L^6(D_t)} \to 0$$

□

We continue in our goal to prove the global existence for the energy critical case. Our next step is to utlize the multiplier method developed by Morawetz. We note

$$(\Box u + u^5)(t\partial_t u + r\partial_r u + u) = 0$$

and we have

$$LHS = \text{ div}(tQ + (\partial_t u)u, -tP) + \frac{1}{3}u^6$$

where

$$Q = \frac{1}{2}|u'|^2 + \frac{1}{6}u^6 + t^{-1}\partial_t u x \cdot \nabla_x u$$

$$P = \left(\frac{1}{2}|\partial_t u|^2 - \frac{1}{2}|\nabla_x u|^2 - \frac{1}{6}u^6\right)\frac{x}{t} + (t^{-1}u + \partial_t u + t^{-1}x \cdot \nabla_x u)\nabla_x u$$

In the above setup shift (T_*, x_0) to $(0,0)$ so that we now consider $t \nearrow 0$. We make the following adjustments to the notation used above:

$$\Lambda(T,S) = \{(t,x) : T \leq t \leq s; |x| \leq -t\}$$

$$D_T = \{(-T,x) : |x| \leq -T\}$$

$$M_T^S = \{(t,x) : T \leq t \leq S; |x| = -T\}$$

Integrating we find

$$-\frac{1}{3}\iint_{\Lambda(T,S)} u^6 \; dxdt = \int_{D_s} sQ + u\partial_t u \; dx - \int_{D_T} TQ + u\partial_t u \; dx + \frac{1}{\sqrt{2}}\iint_{M_T^S} tQ + u\partial_t u + (P\cdot x) \; d\sigma$$

As we let $s \nearrow 0$ we find

$$s\int_{D_s} Q \; dx \to 0$$

because the integral is bounded by energy (for the last Q piece: $\frac{x}{t} \leq 1$ and for $\partial_t u \nabla_x u$ use $ab \leq \frac{1}{2}(a^2 + b^2)$). Furthermore

$$\int_{D_s} u\partial_t u \leq \|u\|_{L^2(D_s)}\|\partial_t u\|_{L^2(D_s)} \lesssim s\|u\|_{L^6(D_s)}\|\partial_t u\|_{L^2(D_s)} \to 0.$$

So

$$-\frac{1}{3}\iint u^6 \; dxdt = -\int_{D_T} TQ + u\partial_t u \; dx + \frac{1}{\sqrt{2}}\iint_{M_T^0} tQ + u\partial_t u + (P\cdot x) \; d\sigma =: I + II$$

Note that it follows that $I + II \leq 0$.
Now we calculate

$$\begin{aligned}
\sqrt{2}II &= \int_{M_T^0} \frac{t}{2}[|\partial_t u|^2 + |\nabla_x u|^2] + \frac{t}{6}u^6 + \partial_t u x \cdot \nabla_x u \\
&\quad + \frac{|x|^2}{2t}\left(|\partial_t u|^2 - |\nabla_x u|^2\right) - \frac{|x|^2}{t}\frac{1}{6}u^6 \\
&\quad + (x \cdot \nabla u)(t^{-1}u + \partial_t u + t^{-1}x \cdot \nabla u) + u\partial_t u \; d\sigma \\
&= \int_{M_T^0} -|x||\partial_t u|^2 + 2\partial_t u(x \cdot \nabla u) - \frac{1}{|x|}(x \cdot \nabla u)^2 - u\left(\frac{x}{|x|} \cdot \nabla u\right) + u\partial_t u \; d\sigma \\
&= -\int_{M_T^0} |x|\left(\frac{x \cdot \nabla_x u}{|x|} - \partial_t u\right)^2 + u\left(\frac{x \cdot \nabla_x u}{|x|} - \partial_t u\right) \; d\sigma
\end{aligned}$$

Parametrize M_T^0 by $y \to (-|y|, y)$ for $|y| \leq |T|$. Then $d\sigma = \sqrt{2}\; dy$, and if we set $v(y) = u(-|y|, y)$ then

$$\frac{y \cdot \nabla_y v}{|y|} = \left(\frac{x}{|x|} \cdot \nabla_x u\right)(-|y|, y) - \frac{y}{|y|}(\partial_t u)(-|y|, y).$$

Therefore,

$$\begin{aligned}
II &= -\int_{|y| \leq |T|} \frac{|y \cdot \nabla v|^2}{|y|} + v\frac{y \cdot \nabla v}{|y|} \; dy \\
&= -\int_{|y| \leq |T|} \frac{1}{|y|}|y \cdot \nabla v + v|^2 \; dy + \int_{|y| \leq |T|} \frac{v^2}{|y|} + v\frac{y \cdot \nabla v}{|y|} \; dy.
\end{aligned}$$

To evaluate the last term we use polar coordinates, which gives $vy \cdot \nabla v/|y| = v\partial_r v = \frac{1}{2}\partial_r(v^2)$. We integrate by parts to find

$$\begin{aligned}
\int_{|y| \leq |T|} v\frac{y \cdot \nabla v}{|y|} \; dy &= \int_{S^2}\int_0^{|T|} \frac{1}{2}\partial_r(v^2)r^2 \; drd\sigma(\omega) \\
&= \frac{1}{2}\int v^2(|t|\omega)|T|^2 \; d\sigma(\omega) - \int_0^{|T|}\int v^2 r \; drd\sigma(\omega) \\
&= \frac{1}{2}\int_{\partial D_T} u^2 \; d\sigma - \int_{|y| \leq T} v^2\frac{dy}{|y|}
\end{aligned}$$

Thus we have

$$II = \frac{1}{\sqrt{2}}\int_{M_T^0} t\left|\frac{x}{|x|} \cdot \nabla u - \partial_t u + \frac{u}{t}\right|^2 \; d\sigma + \frac{1}{2}\int_{\partial D_r} u^2 \; d\sigma.$$

We now consider I. For the integrand in the definition we have

$$
\begin{aligned}
TQ + u\partial_t u &= T(\frac{1}{2}|u'|^2 + \frac{1}{6}u^6) + \partial_t u(u + x\cdot\nabla u) \\
|\partial_t u(u + x\cdot\nabla u)| &\leq \frac{1}{2}\left[(-T)(\partial_t u)^2 + \frac{1}{-T}(u + x\cdot\nabla u)^2\right] \\
&= -T\left[\frac{1}{2}(\partial_t u)^2 + \frac{1}{2}\left(\frac{u}{|x|} + \frac{x}{|x|}\cdot\nabla u\right)^2\right] \\
&\leq -T\left[\frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u + \frac{x}{|x|^2}u|^2\right]
\end{aligned}
$$

It follows that

$$
\begin{aligned}
I &\leq \frac{-T}{6}\int_{D_T} u^6 - T\int_{D_T}\frac{|\nabla u|^2}{2}\,dx + T\int \frac{1}{2}|\nabla u + \frac{x}{|x|^2}u|^2\,dx \\
&= \frac{|T|}{6}\int_{D_T} u^6\,dx + T\int_{D_T} u\frac{x}{|x|^2}\cdot\nabla u\,dx + \frac{T}{2}\int_{D_T}\frac{u^2}{|x|^2}\,dx \\
&= \frac{|T|}{6}\int_{D_T} u^6\,dx - \frac{1}{2}\int_{\partial D_T} u^2\,d\sigma
\end{aligned}
$$

where the last equality follows by computing the same integration by parts as from part II. Thus we have

$$
\begin{aligned}
|T|\int_{D_T}\frac{1}{6}u^6\,dx &\leq I + \frac{1}{2}\int_{\partial D_T} u^2\,d\sigma \\
&\leq -II + \frac{1}{2}\int_{\partial D_T} u^2\,d\sigma \\
&= -\frac{1}{\sqrt{2}}\int_{M_T^0} t\left|\frac{x}{|x|}\cdot\nabla u - \partial_t u + \frac{u}{t}\right|^2\,d\sigma \\
&\leq |T|\int_{M_T^0}\left|\frac{x}{|x|}\cdot\nabla u - \partial_t u\right|\,d\sigma + \int_{M_T^0}\frac{u^2}{|t|}\,d\sigma \\
&\lesssim |T|(\mathrm{Flux}(u, M_T^0)) + \left(\int_{M_T^0} u^6\,d\sigma\right)^{\frac{1}{3}}\left(\int_{M_T^0} t^{-\frac{3}{2}}\right) \\
&\lesssim |T|(\mathrm{Flux}(u, M_T^0)) + (\mathrm{Flux})^{\frac{1}{3}}\left(\int_0^{|T|} t^{\frac{-3}{2}}t^2\,dt\right)^{\frac{2}{3}} \\
&\lesssim |T|(\mathrm{Flux}) + |T|(\mathrm{Flux})^{\frac{1}{3}}
\end{aligned}
$$

Finally we see

$$
\int_{D_t}\frac{1}{6}u^6\,dx \lesssim \mathrm{Flux} + (\mathrm{Flux})^{\frac{1}{3}}
$$

and the RHS goes to 0 as $t \to 0$ so that the LHS does also, as desired.

4 The Strauss Conjecture

Consider the nonlinear wave equation

$$\begin{cases} \Box u = u^p \\ u(0,\cdot) = f \qquad \partial_t u(0,\cdot) = g \end{cases} \tag{18}$$

with $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$. A question we may ask is if f, g are small in the correct sense, how small can p be and still recover global existence? The answer is that there is global existence for $p > p_c$, where $p_c > 1$ solves

$$(n-1)p_c^2 - (n+1)p_c - 2 = 0$$

For $n = 3$ this gives $p_c = 1 + \sqrt{2}$. For $n = 4$ this gives $p_c = 2$. We will discuss a proof by Hidano-Metcalfe-Smith-Sogge-Zhou for $n = 3, 4$. Weighted Strichartz estimates based on a localized energy estimate are key to the proof.

4.1 Localized Energy Estimates ($n \geq 3$)

The localized energy estimate states that if $\Box u = 0$, then

$$\sup_R \left[R^{-\frac{1}{2}} \|u'\|_{L_t^2 L_x^2([0,\infty)\times\{|x|\leq R\})} + R^{-\frac{3}{2}} \|u\|_{L_t^2 L_x^2(\mathbb{R}_+ \times \{|x|<R\})} \right] \lesssim \|u'(0,\cdot)\|_{L^2}$$

We note that by conservation of energy we have

$$\|u'(t,\cdot)\|_{L^2(\mathbb{R}^3)} = \|u'(0,\cdot)\|_{L^2(\mathbb{R}^3)}$$

so that we certainly cannot integrate over time and expect to have a finite quantity. However, if we restrict to a compact spacial region, we are able to obtain decay that allows us to integrate over all time.

Sketch of a proof due to Keel-Smith-Sogge We will do another rigorous proof, but here we note the general idea of a proof for the case where n is odd. By scaling it suffices to take $R = 1$. Let

$$\beta_k = \begin{cases} 1 & k-1 \leq |x| \leq k+2 \\ 0 & |x| \notin [k-2, k+3] \end{cases}$$

and let u_k solve

$$\begin{cases} \Box u_k = 0 \\ u_k(0,\cdot) - \beta_k f \qquad \partial_t u_k(0,\cdot) = \beta_k g \end{cases}$$

Huygens' principle (which relies on the fact that n is odd) then implies $u = u_k$ in $[k, k+1] \times \{|x| \leq 1\}$.

$$
\begin{aligned}
\|u'\|^2_{L^2_t L^2_x([0,\infty)\times\{|x|\le 1\}} &= \sum_{k=0}^{\infty} \|u'\|^2_{L^2_t L^2_x([k,k+1]\times\{|x|\le 1\})} \\
&= \sum_{k=0}^{\infty} \|u'_k\|^2_{L^2_t L^2_x([k,k+1]\times\{|x|\le 1\})} \\
&\lesssim \sum_{k=0}^{\infty} \|u'_k\|^2_{L^\infty L^2([k,k+1]\times\{|x|\le 1\})} \\
&\lesssim \sum_{k=0}^{\infty} \left(\|\nabla(\beta_k f)\|^2_{L^2} + \|\beta_k g\|^2_{L^2}\right) \\
&\lesssim \|\nabla f\|^2_{L^2} + \|g\|^2_{L^2}
\end{aligned}
$$

To see the last inequality notice $(\text{supp}\beta_k \cap \text{ supp}\beta_j) = \emptyset$ if $|j-k| \ge 6$ and $[\sum_k (\beta_k h)^2]^{\frac{1}{2}} \le \sum_k \beta_k h \le 6h$

Proof of weaker estimate using multiplier method The multiplier method involves multiplying $\Box u$ by a cleverly chosen function, taking advantage of the fact that $\Box u = 0$, and integrating by parts. To do the calculations, we first introduce some notation and a decomposition of the gradient into a radial and angular component.

We use the notation

$$
r = |x| \qquad \partial_r u = \frac{x}{r} \cdot \nabla_x u
$$

and decompose the gradient

$$
\nabla u = \frac{x}{r}\partial_r u + \not\nabla u.
$$

We claim that this decomposition is orthogonal (i.e. $\frac{x}{r}\partial_r u \cdot \not\nabla u = 0$). Since $\frac{x}{r}\partial_r u = \frac{x_i \partial_i u}{r}\frac{x}{r}$ it suffices to show $\frac{x}{r} \cdot \not\nabla u = 0$. We calculate

$$
\begin{aligned}
\frac{x}{r} \cdot \not\nabla u &= \frac{x}{r} \cdot \left(\nabla u - \frac{x}{r}\partial_r u\right) \\
&= \frac{x}{r} \cdot \nabla u - \frac{x \cdot x}{r^2}\partial_r u \\
&= \partial_r u - \partial_r u \\
&= 0
\end{aligned}
$$

This orthogonal decomposition gives us the corollary

$$
|\nabla u|^2 = (\partial_r u)^2 + |\not\nabla u|^2.
$$

We will use this in our later calculations.

Now that we have this decomposition, let's find Δ in spherical coordinates:

$$\begin{aligned}
\Delta u &= \nabla \cdot \nabla u \\
&= \nabla \cdot \left(\frac{x}{r} \partial_r u + \not\nabla u \right) \\
&= \left(\nabla \cdot \frac{x}{r} \right) \partial_r u + \frac{x}{r} \cdot \nabla \partial_r u + \nabla \cdot \not\nabla u \\
&= \frac{n-1}{r} \partial_r u + \partial_r^2 u + \left(\frac{x}{r} \partial_r \cdot \not\nabla u + \not\nabla \cdot \not\nabla u \right) \\
&= \frac{n-1}{r} \partial_r u + \partial_r^2 u + \frac{x}{r} \cdot \not\nabla \partial_r u + \frac{x}{r} \left[\partial_r, \not\nabla \right] u + \not\nabla \cdot \not\nabla u
\end{aligned}$$

To simplify the last expression, we find $[\not\nabla_k, \partial_r] u$

$$\begin{aligned}
\not\nabla_k \partial_r u &= \left(\partial_k - \frac{x_k}{r} \partial_r \right) \left(\sum_{j=1}^{n} \partial_j u \right) \\
&= \sum_{j=1}^{n} \partial_k \left(\frac{x_j}{r} \partial_j u \right) - \frac{x_k}{r} \partial_r^2 u \\
&= \sum_{j=1}^{n} \left(\frac{\delta_{jk}}{r} - \frac{x_j x_k}{r^3} \right) \partial_j u + \sum_{j=1}^{n} \frac{x_j}{r} \partial_k \partial_j u - \frac{x_k}{r} \partial_r^2 u \\
&= \frac{1}{r} \left(\partial_k u - \frac{x_k}{r} \partial_r u \right) + \left(\partial_r \partial_k u - \partial_r \left(\frac{x_k}{r} \partial_r u \right) \right) \\
&= \frac{1}{r} \not\nabla_k u + \partial_r \left(\not\nabla_k u \right)
\end{aligned}$$

Thus the commutator is given by

$$[\not\nabla_k, \partial_r] = -\frac{1}{r} \not\nabla_k$$

and we have

$$\begin{aligned}
\Delta u &= \partial_r^2 u + \frac{n-1}{r} \partial_r u + \frac{x}{r} \cdot \not\nabla \partial_r u - \frac{x}{r} \frac{1}{r} \not\nabla u + \not\nabla \cdot \not\nabla u \\
&= r^{-(n-1)} \partial_r (r^{n-1} \partial_r u) + \not\nabla \cdot \not\nabla u
\end{aligned}$$

We are now ready to begin the multiplier method calculations. First we will multiply by $f(r) \partial_r u$, where $f(r)$ is some function. We will determine the necessary properties of f once we've completed the calculations.

$$
\begin{aligned}
0 &= \int_0^t \int \Box u f(r) \partial_r u \; dxdt \\
&= \int_0^t \int \left(\partial_t^2 u - r^{-(n-1)} \partial_r r^{n-1} \partial_r u + \not\nabla \cdot \not\nabla u \right) f(r) \partial_r u \; dxdt \\
&= \int \partial_t u f(r) \partial_r u \; dx \Big|_0^t - \int_0^t \int \partial_t u f(r) \partial_t \partial_r u \; dxdt + \int_0^t \int \partial_r u \partial_r (f(r) \partial_r u) \; dxdt \\
&\quad + \int_0^t \int \not\nabla u \cdot \not\nabla (f(r) \partial_r u) \; dxdt \\
&= \int f(r) \partial_t u \partial_r u \; dx \Big|_0^t - \int_0^t \int f(r) \partial_t u \partial_r \partial_t u \; dxdt + \int_0^t \int f'(r) \left(\partial_r u \right)^2 \; dxdt + \int_0^t \int f(r) \partial_r \partial_r^2 u \; dxdt \\
&\quad \int_0^t \int f(r) \not\nabla u \cdot \not\nabla \partial_r u \; dxdt \\
&= \int f(r) \partial_t u \partial_r u \; dx \Big|_0^t + \frac{1}{2} \int_0^t \int f(r) \left(\partial_r \left[-(\partial_t u)^2 + (partial_r u)^2 + (\not\nabla u)^2 \right] \right) \; dxdt \\
&\quad + \int_0^t \int f'(r) (\partial_r u)^2 + \frac{f(r)}{r} |\not\nabla|^2 \; dxdt \\
&= \int f(r) \partial_t u \partial_r u \; dx \Big|_0^t - \frac{1}{2} \int_0^t \int \left[f'(r) + \frac{n-1}{r} f(r) \right] \left[-(\partial_t u)^2 + |\nabla u|^2 \right] \; dxdt \\
&\quad + \int_0^t \int f'(r) (\partial_r u)^2 + \frac{f(r)}{r} |\not\nabla u|^2 \; dxdt
\end{aligned}
$$

Note that when integrating by parts with respect to r we used $dx = r^{n-1} dr d\sigma$.

We now use the multiplier $\frac{n-1}{2r}f(r)u$:

$$
\begin{aligned}
0 &= \int_0^t \int \Box u \left(\frac{n-1}{2r} f(r) u \right) \, dxdt \\
&= \int_0^t \int \left(\partial_t^2 u - r^{-(n-1)} \partial_r r^{n-1} \partial_r u + \not\nabla \cdot \not\nabla u \right) \frac{n-1}{2} \frac{f(r)}{r} \partial_r u \; dxdt \\
&= \frac{n-1}{2} \int \partial_t u \frac{f(r)}{r} u \; dx \Big|_0^t - \frac{n-1}{2} \int_0^t \int \frac{f(r)}{r} (\partial_t u)^2 \; dxdt + \frac{n-1}{2} \int_0^t \int \partial_r u \partial_r \left(\frac{f(r)}{r} u \right) \; dxdt \\
&\quad + \frac{n-1}{2} \int_0^t \int \frac{f(r)}{r} |\not\nabla u|^2 \; dxdt \\
&= \frac{n-1}{2} \int \partial_t u \frac{f(r)}{r} u \; dx \Big|_0^t + \frac{n-1}{2} \int_0^t \int \frac{f(r)}{r} \left[-(\partial_t u)^2 + |\nabla u|^2 \right] \; dxdt + \frac{n-1}{2} \int_0^t \int u \partial_r u \left(\partial_r \frac{f(r)}{r} \right) \; dxdt \\
&= \frac{n-1}{2} \int \partial_t u \frac{f(r)}{r} u \; dx \Big|_0^t + \frac{n-1}{2} \int_0^t \int \frac{f(r)}{r} \left[-(\partial_t u)^2 + |\nabla u|^2 \right] \; dxdt \\
&\quad - \frac{n-1}{4} \int_0^t \int r^{-(n-1)} \partial_r \left(r^{n-1} \partial_r \left(f(r) r \right) \right) u^2 \; dxdt \\
&= \frac{n-1}{2} \int \partial_t u \frac{f(r)}{r} u \; dx \Big|_0^t + \frac{n-1}{2} \int_0^t \int \frac{f(r)}{r} [-(\partial_t u)^2 + |\nabla u|^2] \; dxdt - \frac{n-1}{4} \int_0^t \int \Delta \left(\frac{f(r)}{r} \right) u^2 \; dxdt
\end{aligned}
$$

Combining these calculations we find

$$
\begin{aligned}
0 &= \int_0^t \int \Box u \left(f(r) \partial_r u + \frac{n-1}{2} \frac{f(r)}{r} u \right) \; dxdt \\
&= \int f(r) \partial_t u \partial_r u \; dx \Big|_0^t + \frac{n-1}{2} \int \frac{f(r)}{r} u \partial_t u \; dx \Big|_0^t + \frac{1}{2} \int_0^t \int f'(r) () \partial_t u)^2 + (\partial_r u)^2) \; dxdt \\
&\quad + \int_0^t \int \left[\frac{f(r)}{r} - \frac{1}{2} f'(r) \right] |\not\nabla u|^2 \; dxdt - \frac{n-1}{4} \int_0^t \int \Delta \left(\frac{f(r)}{r} \right) u^2 \; dxdt
\end{aligned}
$$

We now have an idea of what properties we would like our function $f(r)$ to satisfy. This includes guaranteeing the last three terms are positive. Our wishlist for f is:

- $f \in C^2$
- f bounded
- $f' > 0$
- $\frac{f(r)}{r} - \frac{1}{2} f' > 0$
- $-\Delta \left(\frac{f(r)}{r} \right) > 0$

We note that the fourth item on the list is satisfied if $f > 0$ and $\frac{f}{r} - f' > 0$. One such function is $f(r) = \frac{r}{r+R}$:

$$
f'(r) = \frac{R}{(r+R)^2} \qquad \frac{f(r)}{r} - f'(r) = \frac{1}{1+R} - \frac{1}{r+R} \left(\frac{R}{r+R} \right) > 0
$$

$$-\Delta\left(\frac{f(r)}{r}\right) = \frac{(n-1)R+(n-3)r}{r(r+R)^3} > 0 \quad (n \geq 3)$$

Using this f we find:

$$\begin{aligned}
-\int \frac{r}{r+R}\partial_t u \partial_r u \ dx\Big|_0^t &- \frac{n-1}{2}\int \frac{1}{r+R} u\partial_t u \ dx\Big|_0^t \\
&= \frac{1}{2}\int_0^t \int \frac{R}{(r+R)^2}((\partial_t u)^2 + (\partial_r u)^2) \ dxdt + \int_0^t \int \left[\frac{1}{r+R} - \frac{R}{2(r+R)^2}\right] |\nabla\!\!\!/ u|^2 \ dxdt \\
&\quad + \frac{n-1}{4}\int_0^t \int \frac{(n-1)R+(n-3)r}{r(r+R)^3} u^2 \ dxdt \\
&\gtrsim R^{-1}\int_0^t \int_{|x|\leq R} [(\partial_t u)^2 + (\partial_r u)^2 + |\nabla\!\!\!/ u|^2] \ dxdt + R^{-3}\int_0^t \int_{|x|\leq R} u^2 \ dxdt
\end{aligned}$$

Now we need to show

$$-\int f(r)\partial_t u \partial_r u \ dx\Big|_0^t - \frac{n-1}{2}\int \frac{f(r)}{r} u\partial_t u \ dx\Big|_0^t \lesssim \|u'(0,\cdot)\|_{L^2}^2$$

To do this we use the fact

$$\|u'(t,\cdot)\|_{L^2} = \|u'(0,\cdot)\|_{L^2}$$

and we find

$$\begin{aligned}
\int f(r)\partial_t u(0,\cdot)\partial_r u(0,\cdot) \ dx &\lesssim \|\partial_t u(0,\cdot)\|_{L^2}\|\partial_r u(0,\cdot)\|_{L^2} \\
&\lesssim \|u'(0,\cdot)\|_{L^2} \\
\int \frac{f(r)}{r} u(0,\cdot)\partial_r u(0,\cdot) \ dx &\lesssim \|\frac{1}{r}u(0,\cdot)\|_{L^2}\|u'(0,\cdot)\|_{L^2} \\
&\lesssim \|u'(0,\cdot)\|_{L^2}
\end{aligned}$$

In the last step we use a Hardy inequality:

$$\|\frac{u}{r}\|_{L^2} \lesssim \|\nabla u\|_{L^2} \quad n \geq 3$$

proof of Hardy inequality:

$$\begin{aligned}
\left\|\frac{u}{r}\right\|_{L^2}^2 &= \iint_0^\infty r^{-2}u^2 r^{n-1} \ drd\sigma \\
&= \iint_0^\infty u^2 \frac{1}{n-2}\partial_r(r^{n-2}) \ drd\sigma \\
&= -\frac{2}{n-2}\iint u\partial_r u r^{-1} r^{n-1} \ drd\sigma \\
&\lesssim \left\|\frac{u}{r}\right\|_{L^2} \|\partial_r u\|_{L^2}
\end{aligned}$$

Now everything is bounded by $\|u'\|$, as desired.

Thus we have shown

$$R^{-1}\|u'\|_{L^2}^2 + R^{-3}\|u\|_{L^2}^2 \lesssim \|u'(0,\cdot)\|_{L^2}$$

while the localized energy estimate states

$$R^{-1/2}\|u'\|_{L^2} + R^{-3/2}\|u\|_{L^2} \lesssim \|u'(0,\cdot)\|_{L^2}$$

for all R. Our estimate here just misses.

4.2 Weighted Strichartz Estimates

Trace theorem on S^{n-1}

Theorem 4.1. *Suppose $\frac{1}{2} < s < \frac{n}{2}$. Then we have*

$$\sup_{r>0} r^{\frac{n}{2}-s} \left(\int_{S^{n-1}} |v(r\omega)| \ d\sigma \right)^{1/2} \lesssim \|v\|_{\dot{H}^s}$$

Define the trace map:

$$\mathcal{T} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^{n-1})$$

by

$$\mathcal{T}u = f \quad \text{with } f(x') := \mathcal{T}[u(x_1, x')] = u(0, x')$$

where $x' = (x_2, x_3, \ldots, x_n)$.

Proposition 4.2. *$\mathcal{T}$ extends uniquely to a continuous linear map:*

$$\mathcal{T} : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}) \qquad \textit{for } s > 1/2$$

Proof. Note:

$$\iint g(x_1) e^{-ix_1\xi_1} \ dx_1 d\xi_1 = \int e^{i\xi_1 \cdot 0}[\hat{g}(\xi_1)] \ d\xi_1 = g(0)$$

If $f = \mathcal{T}u$, then $f(x') = u(0, x')$ and we calculate

$$\begin{aligned}
\hat{f}(\xi') &= c \int e^{-ix'\cdot\xi'} f(x') \ dx' \\
&= c \int e^{-ix'\cdot\xi'} u(0, x') \ dx' \\
&= c \iiint e^{-ix'\cdot\xi'} e^{-ix_1\xi_1} u(x_1, x') \ dx_1 dx' d\xi_1 \\
&= c \int \hat{u}(\xi) \ d\xi_1
\end{aligned}$$

Thus we have

$$|\hat{f}(\xi')|^2 \lesssim \left(\int |\hat{u}(\xi)|^2 \langle\xi\rangle^{2s} \ d\xi_1 \right) \left(\int \langle\xi\rangle^{-2s} \ d\xi_1 \right) \lesssim \left(\int |\hat{u}(\xi)|^2 \langle\xi\rangle^{2s} \ d\xi_1 \right) \left(\langle\xi'\rangle^{-2(s-1/2)} \right)$$

Since for $s > 1/2$,

$$\int \langle \xi \rangle^{-2s} \, d\xi_1 = \int \frac{1}{(1+\xi_1^2+|\xi'|^2)^s} \, d\xi_1 = c\frac{1}{(1+|\xi'|^2)^{s-1/2}} = c\langle \xi' \rangle^{-2(s-1/2)}$$

In other words

$$\|f\|^2_{H^{s-1/2}(\mathbb{R}^{n-1})} \approx \int \langle \xi' \rangle^{2(s-1/2)} |\hat{f}(\xi')|^2 \, d\xi' \lesssim \iint |\hat{u}(\xi)|^2 \langle \xi \rangle^{2s} \, d\xi_1 d\xi' \approx \|u\|^2_{H^2}$$

□

Local Energy Estimate
We will use a consequence of the local energy estimate to show

$$\left\| |x|^{-s} e^{it|D|} u \right\|_{L^2_{t,x}} \lesssim \| |D|^{s-1/2} u \|_{L^2_x} \qquad \frac{1}{2} < s < \frac{n}{2} \tag{19}$$

Roughly, we showed in the local energy estimate that for

$$\begin{cases} \Box u = 0 \\ u(0,\cdot) = f, \quad \partial_t u(0,\cdot) = g \end{cases}$$

we have

$$\| |x|^{-1/2} u' \|_{L^2 L^2} \lesssim \| \nabla f \|_{L^2} \qquad \| |x|^{-3/2} u \|_{L^2 L^2} \lesssim \| \nabla f \|_{L^2}$$

(note that we have not proved these estimates - our local energy decay estimates just miss these). Heuristically, we then "interpolate" to find

$$\| |x|^{-s} u \|_{L^2 L^2} \lesssim \| |D|^{s-1/2} f \|_{L^2} \qquad \frac{1}{2} < s < \frac{3}{2}$$

This estimate will be used in our proof of (19).

Proof. [of (19)]

$$\begin{aligned} \left\| |x|^{-s} e^{it|D|} u \right\|^2_{L^2_t L^2_x} &= c \iint \left| |x|^{-s} \int e^{ix\cdot\xi} e^{it|\xi|} \hat{u}(\xi) \, d\xi \right|^2 \, dx dt \\ &= c \iint \left| |x|^{-s} \int e^{ix\cdot\xi} \delta(\tau - |\xi|) \hat{u}(\xi) \, d\xi \right|^2 \, dx d\tau \quad \text{by Plancherel in t} \\ &= c \iint \left| |x|^{-s} \iint_0^\infty e^{ix\cdot\rho\omega} \delta(\tau-\rho) \hat{u}(\rho\omega) \, d\rho d\omega \right|^2 \, dx d\tau \\ &= c \int_0^\infty \int \left| |x|^{-s} \int_{S^{n-1}} e^{ix\cdot\tau\omega} \hat{u}(\tau\omega) \, d\omega \right|^2 \, dx d\tau \end{aligned}$$

We claim that we have by duality

$$\left\| |x|^{-s} \int_{S^{n-1}} h(\omega) e^{i\lambda x\cdot\omega} \, d\sigma \right\|_{L^2(\mathbb{R}^n)} \lesssim \lambda^{s-n/2} \|h\|_{L^2_\omega(S^{n-1})}$$

proof of claim: Recall the trace theorem

$$\left(\int |\hat{q}(\lambda\omega)|^2 \, d\omega\right)^{1/2} \lesssim \lambda^{-\frac{n}{2}+s}\|\hat{w}\|_{\dot{H}^s_\xi} \approx \lambda^{-\frac{n}{2}+s}\||x|^s w\|_{L^2_x}$$

$$\begin{aligned}
\langle |x|^{-s}\int_{S^{n-1}} h(\omega)e^{i\lambda x\omega} \, d\sigma, w\rangle_{L^2_x} &= \int |x|^{-s}\int_{S^{n-1}} h(\omega)e^{i\lambda x\omega} \, d\sigma\hat{w}(x) \, dx \\
&= \int_{S^{n-1}} h(\omega)\overline{\int |x|^{-s}e^{-i\lambda x\omega}w(s) \, dx \, d\sigma} \\
&= \langle h, (|x|^{-s}w)\check{}(\lambda\cdot)\rangle_{L^2_\omega} \\
&\le \|h\|_{L^2_\omega}\|(|x|^{-s}w)\check{}(\lambda\cdot)\|_{L^2_\omega} \\
&\lesssim \|h\|_{L^2_\omega}\lambda^{-\frac{n}{2}+s}\||x|^s|x|^{-s}w\|_{L^2}
\end{aligned}$$

Taking the sup over all w with $\|w\|_{L^2} = 1$ we obtain our desired result.
The claim and the result from the local energy estimate gives us

$$\left\||x|^{-s}e^{it|D|}u\right\|^2_{L^2L^2} \lesssim \int_0^\infty \int \tau^{2(n-1)}\tau^{2s-n}|\hat{u}(\tau\omega)|^2 \, d\sigma d\tau = c\int |\xi|^{2s-1}|\hat{u}(\xi)|^2 \, d\xi = \left\||D|^{s-1/2}u\right\|^2_{L^2}$$

□

4.2.1 Proof of Weighted Strichartz Estimate

We first find for $\frac{1}{2} < s_1 < \frac{n}{2}$

$$\begin{aligned}
\left\||x|^{\frac{n}{2}-s_1}e^{it|D|}\varphi\right\|_{L^\infty_t L^\infty_r L^2_\omega} &\lesssim \left\|e^{it|D|}\varphi\right\|_{L^\infty_t \dot{H}^{s_1}_x} \\
&\approx \left\|e^{it|\xi|}|\xi|^{s_1}\hat{\varphi}(\xi)\right\|_{L^\infty_t L^2_\xi} \\
&\approx \||\xi|^{s_1}\hat{\varphi}(\xi)\|_{L^2_\xi} \\
&\approx \|\varphi\|_{\dot{H}^{s_1}}
\end{aligned}$$

And for $s_2 = s_1 - \frac{1}{2}, 0 < s_s < \frac{n-1}{2}$

$$\left\||x|^{-s_2-\frac{1}{2}}e^{it|D|}\varphi\right\|_{L^2_t L^2_r L^2_\omega} \lesssim \|\varphi\|_{\dot{H}^{s_2}}$$

We then interpolate:

$$\frac{1}{q} = \frac{1-t}{\infty} + \frac{t}{2} \Rightarrow t = \frac{2}{q}$$

for $(1-t)s_1 + ts_2$

$$\left\||x|^{(\frac{n}{2}-s_1)(1-t)+(-\frac{1}{2}-s_2)t}e^{it|D|}\varphi\right\|_{L^q_t L^q_r L^2_\omega} \lesssim \|\varphi\|_{\dot{H}^2}$$

So, $q \ge 2, \frac{1}{2} - \frac{1}{q} < s < \frac{n}{2} - \frac{1}{q} \Rightarrow$

$$\left\||x|^{\frac{n}{2}-\frac{n+1}{q}-s}e^{it|D|}\varphi\right\|_{L^q_t L^q_r L^2_\omega} \lesssim \|\varphi\|_{\dot{H}^s}$$

Thus if

$$\begin{cases} \Box v = 0 \\ v(0,\cdot) = f \qquad \partial_t v(0,\cdot) = g \end{cases}$$

we have

$$\left\| |x|^{\frac{n}{2}-\frac{n+1}{q}-s} v \right\|_{L_t^q L_r^q L_\omega^2} \lesssim \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-1}}$$

4.3 Proof of the Strauss Conjecture

We are concerned with solutions to the equation

$$\begin{cases} \Box u = u^p \\ u(0,\cdot) = \epsilon f \qquad \partial_t u(0,\cdot) = \epsilon g \end{cases}$$

with $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^3$ and $f, g \in C_c^\infty$. For $1+\sqrt{2} < p < 3$, there exists a global solution u if ϵ is sufficiently small.

4.3.1 Background Tools

We have established the weighted Strichartz estimate, $\Box v = 0$ implies

$$\left\| |x|^{\frac{n}{2}-\frac{n+1}{q}-s} v \right\|_{L_t^q L_r^q L_\omega^2} \lesssim \|v(0,\cdot)\|_{\dot{H}^s} + \|\partial_t v(0,\cdot)\|_{\dot{H}^{s-1}}$$

for $2 \le q \le \infty$ and $\frac{1}{2} - \frac{1}{q} < s < \frac{n}{2} - \frac{1}{q}$.
Nonhomogeneous Weighted Strichartz Estimate
Here we state the Strichartz estimate for the nonhomogeneous case. Recall the trace theorem:

$$\sup_{r>0} r^{\frac{n}{2}-s} \|u(r\omega)\|_{L_\omega^2} \lesssim \|u\|_{\dot{H}^s} \qquad \frac{1}{2} < s < \frac{n}{2}$$

Dual to this estimate we have

$$\|\varphi\|_{\dot{H}^{-\gamma}} \lesssim \left\| |x|^{-\frac{n}{2}+\gamma} \varphi \right\|_{L_r^1 L_\omega^2} \qquad \frac{1}{2} < \gamma < \frac{n}{2}$$

$$\|\varphi\|_{\dot{H}^{s-1}} \lesssim \left\| |x|^{-\frac{n}{2}+1-s} \varphi \right\|_{L_r^1 L_\omega^2} \qquad \frac{1}{2} < 1-s < \frac{n}{2}$$

Combining this with Duhamel we obtain for $2 \le q \le \infty$, $\frac{1}{2} - \frac{1}{q} < s < \frac{n}{2} - \frac{1}{q}$, and $\frac{1}{2} < 1-s < \frac{n}{2}$

$$\left\| |x|^{\frac{n}{2}-\frac{n+1}{q}-s} u \right\|_{L_t^q L_r^q L_\omega^2} \lesssim \|u(0,\cdot)\|_{\dot{H}^s} + \|\partial_t u(0,\cdot)\|_{\dot{H}^{s-1}} + \||x|^{-\frac{n}{2}+1-s} \Box u\|_{L_t^1 L_r^1 L_\omega^2} \tag{20}$$

Sobolev embeddings on S^2
Set $\Omega_{ij} = x_i \partial_j - x_j \partial_i$ with $1 \le i, j \le n$ and $\Omega = \{\Omega_{ij}\}$.
For $n = 3$ we have

$$\|f(\omega)\|_{L^\infty(S^2)} \lesssim \sum_{|\alpha| \le 2} \|\Omega^\alpha f\|_{L^2(S^2)}$$

$$\|f(\omega)\|_{L^4(S^2)} \lesssim \sum_{|\alpha|\leq 1} \|\Omega^\alpha f(\omega)\|_{L^2(S^2)}$$

Some Key Facts We state but do not calculate $[\Box, \Omega] = 0$. That is, $\Box$ and Ω commute. Thus we have

$$\begin{cases} \Box u = F \\ u(0,\cdot) = f \quad \partial_t(0,\cdot) = g \end{cases} \Rightarrow \begin{cases} \Box \Omega u = \Omega F \\ \Omega u(0,\cdot) = \Omega f \quad \Omega \partial_t u(0,\cdot) = \Omega g \end{cases}$$

Taking care of numerology, which will be used to rewrite (20), we take

$$s = \frac{3}{2} - \frac{2}{p-1} \qquad -\alpha = \frac{3}{2} - \frac{4}{p} - s \quad q = p$$

Thus $-\frac{3}{2} + 1 - s = -p\alpha$. Considering the requirements for (20) to hold, we find

$$\frac{1}{2} - \frac{1}{p} < s \Rightarrow p > 1 + \sqrt{2} \qquad \frac{1}{2} < 1 - s \Rightarrow p < 3$$

With this numerology, we rewrite (20) to find

$$\||x|^{-\alpha} u\|_{L^pL^pL^2} \lesssim \|u(0,\cdot)\|_{\dot{H}^s} + \|\partial_t u(0,\cdot)\|_{\dot{H}^{s-1}} + \||x|^{-p\alpha}\Box u\|_{L^1L^1L^2}$$

4.3.2 Proof

We will use the notation $Z = \{\nabla, \Omega\}$ and $\|Z^{\leq k} f\| = \sum_{|\alpha|\leq k} \|Z^{\alpha f}\|$.

Proof. Take $u_{-1} \equiv 0$ and define u_j to solve

$$\begin{cases} \Box u_j = u_{j-1}^p \\ u_u(0,\cdot) = \epsilon f; \quad \partial_t u_j(0,\cdot) = \epsilon g \end{cases}$$

Step 1 Boundedness.

We wish to show

$$\||x|^{-\alpha} Z^{\leq 2} u_j\|_{L^pL^pL^2} \leq 2C_1\epsilon$$

We have

$$\||x|^{-\alpha} Z^{\leq 2} u_0\|_{L^pL^pL^2} \leq C\left(\epsilon\|Z^{\leq 2} f\|_{\dot{H}^s} + \epsilon\|Z^{\leq 2} g\|_{\dot{H}^{s-1}}\right) \leq C_1\epsilon$$

Arguing inductively, we find

$$\||x|^{-\alpha} Z^{\leq 2} u_j\|_{L^pL^pL^2} \leq C_1\epsilon + C\||x|^{-p\alpha} Z^{\leq 2} u_{j-1}^p\|_{L^1L^1L^2}$$

$$|Z^{\leq 2} u_{j-1}^p| \lesssim |u_{j-1}|^{p-1}|Z^{\leq 2} u_{j-1}| + |u_{j-1}|^{p-2}|Z^{\leq 1} u_{j-1}|^2$$

$$\|Z^{\leq 2} u_{j-1}^p\|_{L^2_\omega} \lesssim \|u_{j-1}\|^{p-1}_{L^\infty_\omega}\|Z^{\leq 2} u_{j-1}\|_{L^2_\omega} + \|u_{j-1}\|^{p-2}_{L^\infty_\omega}\|Z^{\leq 1} u_{j-1}\|^2_{L^4_\omega} \lesssim \|Z^{\leq 2} u_{j-1}\|^p_{L^2_\omega}$$

Plugging in the above calculations gives

$$\begin{aligned} \||x|^{-\alpha} Z^{\leq 2} u_j\|_{L^pL^pL^2} &\leq C_1\epsilon + C\||x|^{-p\alpha}\|Z^{\leq 2} u_{j-1}\|_{L^2_\omega}\|^2\|_{L^1_tL^1_r} \\ &\leq C_1\epsilon + C\||x|^{-\alpha} Z^{\leq 2} u_{j-1}\|^p_{L^pL^pL^2} \\ &\leq C_1\epsilon + C(2C_1]epsilon)^p \\ &\leq 2C_1\epsilon \end{aligned}$$

where the 3rd line follows from the induction hypothesis and the 4th line follows for small enough ϵ.

Step 2 Convergence.

We wish to show

$$\||x|^{\alpha}(u_j - u_{j-1})\|_{L^pL^pL^2} \leq \frac{1}{2}\||x|^{-\alpha}(u_{j-1} - u_{j-2})\|_{L^pL^pL^2}$$

Note that

$$\Box(u_j - u_{j-1}) = u_{j-1}^p - u_{j-2}^p = O(|u_j - 1|^{p-1} + |u_{j-2}|^{p-1})(u_{j-1} - u_{j-2})$$

and we have vanishing data.

$$\begin{aligned}
\||x|^{\alpha}(u_j - u_{j-1})\|_{L^pL^pL^2} &\leq C\||x|^{-p\alpha}O(|u_j - 1|^{p-1} + |u_{j-2}|^{p-1})|u_{j-1} - u_{j-2}|\|_{L^1L^1L^2} \\
&\leq C\left[\||x|^{-\alpha}u_{j-1}\|_{L^pL^pL^\infty}^{p-1} + \||x|^{-\alpha}u_{j-2}\|_{L^pL^pL^\infty}^{p-1}\right]\||x|^{-\alpha}|u_{j-1} - u_{j-2}|\|_{L^pL^pL^2} \\
&\leq C\left[\||x|^{-\alpha}Z^{\leq 2}u_{j-1}\|_{L^pL^pL^2}^{p-1} + \||x|^{-\alpha}Z^{\leq 2}u_{j-2}\|_{L^pL^pL^2}^{p-1}\right]\||x|^{-\alpha}|u_{j-1} - u_{j-2}|\|_{L^pL^pL^2} \\
&\leq 2C(2C_1\epsilon)^{p-1}\||x|^{-\alpha}|u_{j-1} - u_{j-2}|\|_{L^pL^pL^2} \\
&\leq \frac{1}{2}\||x|^{-\alpha}(u_{j-1} - u_{j-2})\|_{L^pL^pL^2}
\end{aligned}$$

where the last line follows if ϵ is small enough.

□

5 Exterior Domains

Let κ be a region such that $0 \in \kappa \subseteq \{|x| < 1\}$ with smooth boundary and assume κ is star shaped with respect to 0. The star shaped assumption is equivalent to $\vec{x} \cdot \vec{\nabla}\Big|_{\partial\kappa} \geq 0$. We consider the equation

$$\begin{cases} \Box u = F \\ u|_{\partial\kappa} = 0 \\ u(0, \cdot) = f; \quad \partial_t u(0, \cdot) = g \end{cases}$$

Much of what we have done before is no longer applicable after introducing the boundary. For example, the Fourier Transform technique is a nightmare becuase of boundary values.

5.1 Local Energy Estimate

We have the energy bound

$$\|u'(t, \cdot)\|_{L^2(\mathbb{R}^3\backslash\kappa)}^2 \lesssim \|u'(0, \cdot)\|_{L^2(\mathbb{R}^3\backslash\kappa)}^2 + \int_0^t \int_{\mathbb{R}^3\backslash\kappa} |\Box u||\partial_t u| \; dxdt$$

Proof.

$$\begin{aligned}
\int_0^t \int \Box u \partial_t u \; dxdt &= \int_0^t \int (\partial_t^2 - \nabla \cdot \nabla) u \partial_t u \; dxdt \\
&= \int_0^t \int \frac{1}{2} \partial_t (\partial_t u)^2 \; dxdt - \int_0^t \int \nabla \cdot (\nabla u \partial_t u) \; dxdt + \int_0^t \int \nabla u \cdot \nabla \partial_t u \; dxdt \\
&= \int_0^t \int \frac{1}{2} \partial_t \left[(\partial_t u)^2) + |\nabla u|^2 \right] \; dxdt - \int_0^t \int \nabla \cdot (\nabla u \partial_t u) \; dxdt \\
&= \frac{1}{2} \|u'(s, \cdot)\|_{L^2}^2 \Big|_0^t + \int_0^t \int_{\partial \kappa} \nabla u \partial_t u \; dxdt \\
&= \frac{1}{2} \|u'(s, \cdot)\|_{L^2}^2 \Big|_0^t
\end{aligned}$$

□

We establish the local energy estimate as before.

$$\begin{aligned}
\int_0^t \int \Box u f(r)\partial_r u \; dxdt &= \int_0^t \int (\partial_t^2 u - \nabla\cdot\nabla u) f(r)\partial_r u \; dxdt \\
&= \int f(r)\partial_t u \partial_r u \; dx\Big|_0^t - \int_0^t \int \partial_t u f(r)\partial_t\partial_r u \; dxdt - \int_0^t \int \nabla\cdot(\nabla u f(r)\partial_r u) \; dxdt \\
&\quad + \int_0^t \int \nabla u\cdot\nabla(f(r)\partial_r u) \; dxdt \\
&= \int f(r)\partial_t u \partial_r u \; dx\Big|_0^t - \int_0^t \int \partial_t u f(r)\partial_t\partial_r u \; dxdt - \int_0^t \int \nabla\cdot(\nabla u f(r)\partial_r u) \; dxdt \\
&\quad + \int_0^t \int \nabla u\cdot\left(\frac{x}{r}f'(r)\partial_r u + f(r)\partial_r\nabla u + \frac{f(r)}{r}\not\nabla u\right) \; dxdt \\
&= \int f(r)\partial_t u \partial_r u \; dx\Big|_0^t + \int_0^t \int \frac{1}{2}f(r)\partial_r\left[-(\partial_t u)^2 + |\nabla u|^2\right] \; dxdt \\
&\quad + \int_0^t \int f'(r)(\partial_r u)^2 + \frac{f(r)}{r}|\not\nabla u|^2 \; dxdt - \int_0^t \int \nabla\cdot(\nabla u f(r)\partial_r u) \; dxdt \\
&= \int f(r)\partial_t u \partial_r u \; dx\Big|_0^t + \int_0^t \int \nabla\cdot\left(\frac{1}{2}f(r)\frac{x}{r}\left[-(\partial_t u)^2 + |\nabla u|^2\right]\right) \; dxdt \\
&\quad - \int_0^t \int \frac{1}{2}\left[\nabla\cdot\left(f(r)\frac{x}{r}\right)\right]\left[-(\partial_t u)^2 + |\nabla u|^2\right] \; dxdt \\
&\quad + \int_0^t \int f'(r)(\partial_r u)^2 + \frac{f(r)}{r}|\not\nabla u|^2 \; dxdt - \int_0^t \int \nabla\cdot(\nabla u f(r)\partial_r u) \; dxdt \\
&= \int f(r)\partial_t u \partial_r u \; dx\Big|_0^t + \int_0^t \int f'(r)(\partial_r u)^2 + \frac{f(r)}{r}|\not\nabla u|^2 \; dxdt \\
&\quad - \frac{1}{2}\int_0^t \int \left[f'(r) + \frac{(n-1)f(r)}{r}\right]\left[-(\partial_t u)^2 + |\nabla u|^2\right] \; dxdt \\
&\quad + \frac{1}{2}\int_0^t \int_{\partial\kappa} \vec{\nabla}\cdot\frac{x}{r}f(r)\left[-(\partial_t u)^2 + |\nabla u|^2\right] \; d\sigma dt + \int_0^t \int_{\partial\kappa} \nabla\cdot\nabla u f(r)\partial_r u \; d\sigma dt
\end{aligned}$$

We note that $\nabla\cdot\frac{x}{r} \geq 0$ and $\partial_t u|_{\partial\kappa} = 0$ so that the second to last term is nonnegative. Furthermore

$$\begin{aligned}
\nabla\cdot\nabla u f(r)\partial_r u|_{\partial\kappa} &= f(r)\partial_{\nabla} u\frac{x}{r}\nabla u|_{\partial\kappa} \\
&= f(r)\frac{x}{r}\cdot\nabla(\partial_{\nabla} u)^2|_{\partial\kappa} \geq 0
\end{aligned}$$

so that the last term is nonnegative. Thus we have

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}^n\setminus\kappa} \Box u f(r)\partial_r u \; dxdt \geq{}& \int f(r)\partial_t u \partial_r u \; dx\Big|_0^t + \frac{1}{2}\int_0^t \int f'(r)\left[(\partial_t u)^2 + (\partial_r u)^2\right] \; dxdt \\
&+ \int_0^t \int \left[\frac{f(r)}{r} - \frac{1}{2}f'(r)\right]|\not\nabla u|^2 \; dxdt \\
&+ \frac{n-1}{2}\int_0^t \int \left[\frac{f(r)}{r}\right]\left((\partial_t u)^2 - |\nabla u|^2\right) \; dxdt
\end{aligned}$$

We now consider the multiplier $\frac{n-1}{2}\frac{f(r)}{r}u$:

$$\begin{aligned}
\int_0^t \int \Box u \frac{n-1}{2}\frac{f(r)}{r}u \; dxdt &= \int_0^t \int (\partial_t^2 - \nabla\cdot\nabla) u \frac{n-1}{2}\frac{f(r)}{r} u \; dxdt \\
&= \frac{n-1}{2}\int \partial_t u \frac{f(r)}{r} u \; dx\Big|_0^t - \int_0^t \int \frac{n-1}{2}\frac{f(r)}{r}(\partial_t u)^2 \; dxdt \\
&\quad - \frac{n-1}{2}\int_)^t \int \nabla\cdot\left(\nabla u \frac{f(r)}{r}u\right) \; dxdt + \frac{n-1}{2}\int_0^t \int \nabla u \cdot \nabla\left(\frac{f(r)}{r}u\right) \; dxdt \\
&= \frac{n-1}{2}\int \partial_t u \frac{f(r)}{r} u \; dx\Big|_0^t - \int_0^t \int \frac{n-1}{2}\frac{f(r)}{r}(\partial_t u)^2 \; dxdt \\
&\quad - \frac{n-1}{2}\int_0^t \int_{\partial\kappa} \nabla\cdot\nabla u \frac{f(r)}{r} u \; d\sigma dt + \frac{n-1}{4}\int_0^t \int \nabla\left(\frac{f(r)}{r}\right)\cdot\nabla(u^2) \\
&\quad + \frac{n-1}{2}\int_0^t \int \frac{f(r)}{r}|\nabla u|^2 \; dxdt \\
&= \frac{n-1}{2}\int \partial_t u \frac{f(r)}{r} u \; dx\Big|_0^t + \frac{n-1}{2}\int_0^t \int \frac{f(r)}{r}\left[-(\partial_t u)^2 + |\nabla u|^2\right] \; dxdt \\
&\quad + \frac{n-1}{4}\int_0^t \int_{\partial\kappa} \nabla\cdot\left[\nabla\left(\frac{f(r)}{r}\right)u^2\right] \; d\sigma dt - \frac{n-1}{4}\int_0^t \int \left[\Delta\left(\frac{f(r)}{r}\right)\right] u^2 \; dxdt
\end{aligned}$$

Combining our results we obtain

$$\begin{aligned}
\int_0^t \int \Box u \left(f(r)\partial_r u + \frac{n-1}{2}\frac{f(r)}{r}u\right) \; dxdt \geq& \int f(r)\partial_r u \partial_t u \; dx\Big|_0^t + \frac{n-1}{2}\int \partial_t u \frac{f(r)}{r} u \; dx\Big|_0^t \\
&+ \int_0^t \int \frac{1}{2}f'(r)\left[(\partial_r u)^2 + (\partial_t u)^2\right] \; dxdt \\
&+ \int_0^t \int \left[\frac{f(r)}{r} - \frac{1}{2}f'(r)\right] |\nabla\!\!\!/ u|^2 \; dxdt \\
&- \frac{n-1}{4}\int_0^t \int \Delta\left(\frac{f(r)}{r}\right) u^2 \; dxdt
\end{aligned}$$

Taking $f(r) = \frac{r}{r+2^j}$ we argue as before to obtain

$$\|u'\|_{L_t^\infty L_x^2} + \sup_{j\geq 0}\left[2^{-j/2}\|u'\|_{L_t^2 L_x^2(\mathbb{R}_+\times\langle x\rangle\approx 2^j)} + 2^{-3j/2}\|u\|_{L_t^2 L_x^2(\mathbb{R}_+\times\langle x\rangle\approx 2^j)}\right] \lesssim \|\nabla f\|_{L^2} + \|g\|_{L^2} + \int_0^\infty \|F(s,\cdot)\|_{L^2} \; ds$$

5.2 Weighted Strichartz Estimate Revisited

Our goal is to obtain the weighted Stricharz estimate for exterior domains. That is we wish to obtain that for $2 \leq p \leq \infty$, $\frac{1}{2} - \frac{1}{p} < \gamma < \frac{n}{2} - \frac{1}{p}$, $\frac{1}{2} < 1 - \gamma < \frac{n}{2}$ we have

$$\||x|^{n/2-(n+1)/p-\gamma}u\|_{L_t^p L_r^p L_\omega^2} \lesssim \|u(0,\cdot)\|_{\dot{H}^\gamma} + \|\partial_t u(0,\cdot)\|_{\dot{H}^{\gamma-1}} + \||x|^{-n/2+1-\gamma}\Box u\|_{L_t^1 L_r^1 L_\omega^2}$$

We note that combining this result with Duhamel gives that if $\Box u = F_1 + F_2$, then

$$\||x|^{n/2-(n+1)/p-\gamma}u\|_{L^p_tL^p_rL^2_\omega} \lesssim \|u(0,\cdot)\|_{\dot{H}^\gamma}+\|\partial_t u(0,\cdot)\|_{\dot{H}^{\gamma-1}}+\||x|^{-n/2+1-\gamma}F_1\|_{L^1_tL^1_rL^2_\omega}+\int_0^\infty \|F_2(s,\cdot)\|_{\dot{H}^{\gamma-1}}\, ds$$

Proposition 5.1. *If w is a solution to*

$$\begin{cases} \Box w = F \\ w(0,\cdot) = 0 = \partial_t w(0,\cdot) \\ supp\ F \subseteq \{|x| \leq 10\} \end{cases}$$

Then for $p > 2, \gamma \geq -\frac{n-3}{2}$

$$\||x|^{n/2-(n+1)/p-\gamma}w\|_{L^p_tL^p_rL^2_\omega} \lesssim \|F\|_{L^2_tH^{\gamma-1}_x}$$

Before proving the proposition, we state and prove the following lemma due to Smith and Sogge.

Lemma 5.2. *Let $\beta \in C^\infty_c(\mathbb{R}^n)$ and $0 \leq \gamma \leq \frac{n-1}{2}$. Then*

$$\int_{-\infty}^\infty \|\beta(\cdot)\left(e^{it|D|}f\right)(t, cdot)\|^2_{H^\gamma}\ dt \lesssim \|f\|^2_{\dot{H}^\gamma}$$

Note that $e^{it|D|}f \approx u$ where u solves $\Box u = 0$, $u(0,\cdot) = f$, $\partial_t u(0,\cdot) = g$.

Proof. LHS:

$$\iint (1+|\xi|^2)^\gamma \left|\int \hat{\beta}(\xi-\eta)e^{it|\eta|}\hat{f}(\eta)\ d\eta\right|^2\ d\xi dt$$
$$= \iint (1+|\xi|^2)^\gamma \left|\int \hat{\beta}(\xi-\eta)\delta(\tau-|\eta|)\hat{f}(\eta)\ d\eta\right|^2\ d\xi d\tau$$
$$\leq \iint (1+|\xi|^2)^\gamma \left[\int |\hat{\beta}(\xi-\eta)|\delta(\tau-|\eta|)\ d\eta\right]\left[\int |\hat{\beta}(\xi-\eta)|\delta(\tau-|\eta|)|\hat{f}(\eta)|^2\ d\eta\right]\ d\xi d\tau$$

It is left as an exercise to show that for $0 \leq \gamma \leq \frac{n-1}{2}$

$$\sup_\xi (1+|\xi|^2)^\gamma \left[\int |\hat{\beta}(\xi-\eta)|\delta(\tau-|\eta|)\ d\eta\right] \leq C\min[\tau^{n-1}, (1+\tau^2)^\gamma]$$

We then obtain

$$\begin{aligned} LHS &\leq \iiint |\hat{\beta}(\xi-\eta)|\delta(\tau-|\eta|)|\hat{f}(\eta)|^2\ d\eta\left(\min[\tau^{n-1},(1+\tau^2)^\gamma]\right)\ d\xi d\tau \\ &\lesssim \iint \delta(\tau-|\eta|)|\hat{f}(\eta)|^2\tau^{2\gamma}\ d\tau d\eta \\ &= \int |\hat{f}(\eta)|^2|\eta|^{2\gamma}\ d\eta \\ &= \|f\|^2_{\dot{H}^\gamma} \end{aligned}$$

□

We are now ready to prove Prop 5.1

Proof. Writing down Duhamel's formula and replacing t by ∞, the Christ-Kiselev lemma implies that is suffices to show

$$\left\| |x|^{n/2-(n+1)/p-\gamma} \int_0^\infty e^{i(t-s)|D|} |D|^{-1} \beta(\cdot) F(s,\cdot)\, ds \right\|_{L^p L^p L^2} \lesssim \|F\|_{L^2_t H^{\gamma-1}_x}$$

From the weighted strichartz estimate we have

$$LHS = \left\| |x|^{n/2-(n+1)/p-\gamma} e^{it|D|} |D|^{-1} \int_0^\infty e^{-is|D|} \beta(\cdot) F(s,\cdot)\, ds \right\|_{L^p L^p L^2} \lesssim \left\| \int_0^\infty e^{-is|D|} \beta(\cdot) F(s,\cdot)\, ds \right\|_{\dot{H}^{\gamma-1}}$$

Thus it suffices to show

$$\left\| \int_0^\infty e^{-is|D|} |D|^{\gamma-1} \beta(\cdot) (1-\Delta)^{\frac{1-\gamma}{2}} H(s,\cdot)\, ds \right\|_{L^2} \lesssim \|H\|_{L^2 L^2}$$

and take $H = (1-\Delta)^{\frac{\gamma-1}{2}} F$ to get what we want. Duality implies that this is equivalent to

$$\left\| (1-\Delta)^{\frac{1-\gamma}{2}} \beta(\cdot) e^{is|D|} |D|^{\gamma-1} h \right\|_{L^2 L^2} \lesssim \|h\|_{L^2} \Leftrightarrow \left\| (1-\Delta)^{\frac{1-\gamma}{2}} \beta(\cdot) e^{is|D|} f \right\|_{L^2 L^2} \lesssim \| |D|^{1-\gamma} f\|_{L^2}$$

which is precisely the lemma if $1-\gamma \le \frac{n-1}{2}$ (i.e. $\gamma \ge -\frac{n-3}{2}$). □

5.3 Global Existence for Small Data

Let $1+\sqrt{2} < p < 3$ and consider the equation

$$\begin{cases} \Box u = u^p \\ u|_{\partial\kappa} = 0 \\ u(0,\cdot) = \epsilon f; \quad \partial_t u(0,\cdot) = \epsilon g \end{cases} \tag{21}$$

where $(t,x) \int \mathbb{R}_+ \times \mathbb{R}^3 \setminus \kappa$, $f, g \in C_c^\infty$, and κ satisfies the following:

1. $0 \in \kappa \subseteq \{|x| < 1\}$ with smooth boundary
2. κ is star shaped wrt 0

Our goal is to show that if ϵ is small enough, there exists a global solution u.

5.3.1 A Useful Consequence of Weighted Strichartz Estimates

We will establish a bound using the weighted Strichraz estimates on solutions to the linear nonhomogenous equation on an exterior domain:

$$\begin{cases} \Box u = F \\ u|_{\partial\kappa} = 0 \\ u(0,\cdot) = \epsilon f; \quad \partial_t u(0,\cdot) = \epsilon g \end{cases} \tag{22}$$

Fix

$$\beta = \begin{cases} 0 & on |x| < 1 \\ 1 & on |x| > 5 \end{cases}$$

Note that if we consider $\Box \beta u$ it is now boundaryless since our function does not see the boundary of κ. Thus our work on $\mathbb{R}^3$ is applicable. We again use the notation $Z = \{\partial_\alpha, \Omega_{ij} = x_i\partial_j - x_j\partial_i\}$. Commuting $\Box$ with $\beta Z^{\leq 2}$ we have

$$\begin{aligned} \Box \beta Z^{\leq 2} u &= \beta Z^{\leq 2} F + [\Box, \beta] Z^{\leq 2} u \\ &= \beta Z^{\leq 2} F - 2\nabla \beta \cdot \nabla Z^{\leq 2} u - (\Delta \beta) Z^{\leq 2} u \\ &= \beta^p Z^{\leq 2} F + (\beta - \beta^p) Z^{\leq 2} F - 2\nabla\beta \cdot \nabla Z^{\leq 2} u - (\Delta\beta) Z^{\leq 2} u \end{aligned}$$

By the weighted Strichartz estimates,

$$\begin{aligned} \| |x|^{-\alpha} \beta Z^{\leq 2} u \|_{L^p L^p L^2} &\lesssim \epsilon + \| |x|^{-p\alpha} \beta^p Z^{\leq 2} F \|_{L^1 L^1 L^2} + \| (\beta - \beta^p) z^{\leq 2} F - 2\nabla\beta \cdot \nabla z^{\leq 2} u - (\Delta\beta) Z^{\leq 2} u \|_{L^2 H^{\gamma - 1}} \\ &\lesssim \epsilon + \| |x|^{-p\alpha} \beta^p Z^{\leq 2} F \|_{L^1 L^1 L^2} + \| \partial^{\leq 2} F \|_{L^2 L^2 (1 \leq x \leq 5)} + \| \partial^{\leq 2} (u, \partial u) \|_{L^2 L^2 (1 \leq x \leq 5)} \end{aligned}$$

Note that for initial data $u(0,\cdot) = f$ and $\partial_t(0,\cdot) = g$, the definition of $\Box$ gives us for the inhomogeneous equation

$$\partial_t^2 u(0,\cdot) = \Delta u(0,\cdot) + F(0,\cdot) = \Delta f + F(0,\cdot)$$

This tells us we need compatibility conditions. For example:

$$0 = \partial_t^2 u(0,\cdot)|_{\partial\kappa} = \Delta f|_{\partial\kappa} + F(0,\cdot)|_{\partial\kappa}$$
$$0 = \partial_t^3 u(0,\cdot)|_{\partial\kappa} = \Delta \partial_t u(0,\cdot)|_{\partial\kappa} + \partial_t F(0,\cdot)|_{\partial\kappa} = \Delta g|_{\partial\kappa} + \partial_t F(0,\cdot)|_{\partial\kappa}$$

We wish to obtain bounds on the L^2L^2 norm of u and it's gradient in the RHS of the above estimate. We will also establish bounds on the $L^\infty L^2$ norm of up to two derivatives of the gradient of u.

Recall the local energy estimate

$$\|u'\|_{L_t^\infty L_x^2} + \sup_R \left[R^{-\frac{1}{2}} \|u'\|_{L_t^2 L_x^2([0,\infty)\times\{|x|\leq R\})} + R^{-\frac{3}{2}} \|u\|_{L_t^2 L_x^2(\mathbb{R}_+ \times \{|x|<R\})} \right] \lesssim \|u'(0,\cdot)\|_{L^2} + \|F\|_{L_t^1 L_x^2}$$

To find the desired bounds, we want to introduce $\partial^{\leq 2}$. However, ∂_x does not preserve the Dirichlet boundary condition, so we do not have that $\partial_x u$ solves our equation. However ∂_t does preserve the boundary conditions. We replace u by $\partial^{\leq 2} u$ in (22) to obtain

$$\|\partial_t^{\leq 2} u'\|_{L_t^\infty L_x^2} + \sup_R \left[R^{-\frac{1}{2}} \|\partial_t^{\leq 2} u'\|_{L_t^2 L_x^2([0,\infty)\times\{|x|\leq R\})} + R^{-\frac{3}{2}} \|\partial^{\leq 2} u\|_{L_t^2 L_x^2(\mathbb{R}_+ \times \{|x|<R\})} \right] \lesssim \epsilon + \|\partial_t^{\leq 2} F\|_{L_t^1 L_x^2}$$

This provides partial results toward our goal, but we still need to handle terms with spatial derivatives.

We use elliptic regularity for the spatial derivatives. For $\mathbb{R}^3$ we have

$$\sum_{i,j=1}^3 \|\partial_{x_i}\partial_{x_j} u\|_{L^2(\mathbb{R}^3)}^2 = \sum_{i,j=1}^3 \int (\partial_{x_i}\partial_{x_j} u)(\partial_{x_i}\partial_{x_j} u)\ dx = \sum_{i,j=1}^3 \int \partial_{x_i}^2 u \partial_{x_j}^2 u\ dx$$

and

$$\|\partial_{x_i}\partial_{x_j}u\|_{L^2} \approx \|\xi_i\xi_j\hat{u}\|_{L^2} \lesssim \||\xi|^2\hat{u}\|_{L^2} \approx \|\Delta u\|_{L^2}$$

Thus we see the L^2 norm of the sum of second order spatial derivatives is approximately the L^2 norm of the spacial Laplacian of u. It follows that

$$\|\partial_x^{\leq 2}u\|_{L^2(\mathbb{R}^3\backslash\kappa)} \lesssim \|\Delta u\|_{L^2(\mathbb{R}^3\backslash\kappa)} + \|\partial^{\leq 1}u\|_{L^2(\mathbb{R}^3\backslash\kappa)}$$

We now turn our attention to $\|\partial u'\|_{L^\infty L^2}$, which we write

$$\|\partial u'\|_{L^\infty L^2} = \|\partial_t u'\|_{L^\infty L^2} + \|\partial_x^{=2}u\|_{L^\infty L^2}$$

The first term is handled by the local energy estimate above. For the second term we use elliptic regularity as above:

$$\begin{aligned}
\|\partial_x^{=2}u\|_{L^\infty L^2} &\lesssim \|\Delta u\||_{L^\infty L^2} + \|\partial^{\leq 1}u|_{L^\infty L^2} \\
&\lesssim \|\partial_t^2 u\||_{L^\infty L^2} + \|F\||_{L^\infty L^2} + \|\partial u\||_{L^\infty L^2} \\
&\lesssim \epsilon + \|\partial_t^{\leq 1}F\|_{L^1L^2} + \|F\||_{L^\infty L^2}
\end{aligned}$$

In a similar fashion we handle $\|\partial^{=2}u'\||_{L^\infty L^2}$ by separating the spatial and time derivatives. We only have left to bound the piece with 3 spatial derivates:

$$\begin{aligned}
\|\partial_x^2\partial_x u\||_{L^\infty L^2} &\lesssim \|\Delta\partial_x u\||_{L^\infty L^2} + \text{ lower order terms} \\
&\lesssim \|\partial_t^2\partial_x u\||_{L^\infty L^2} + \partial^{\leq 1}F\||_{L^\infty L^2}
\end{aligned}$$

Since we have already obtained a bound on the first term on the RHS, and we allow the second term to appear in our estimate, we are done.

Combining the above results gives for a solution u of (22)

$$\begin{aligned}
\||x|^{-\alpha}\beta Z^{\leq 2}u\|_{L^pL^pL^2} &+ \|\partial^{\leq 2}u'\||_{L^\infty L^2} + \|\partial^{\leq 2}u\||_{L^2L^2} \\
&\lesssim \epsilon + \||x|^{-\alpha}\beta^p Z^{\leq 2}u\|_{L^1L^1L^2} + \|\partial^{\leq 2}F\|_{L^1L^2} + \|\partial_{\leq 1}F\|_{L^\infty L^2} + \|\partial^{\leq 2}F\|_{L^2L^2}
\end{aligned}$$

For the sake of simplicity, we note that the last two terms are easier to handle than the third term, so we simply drop them. We will use this estimate in our sketch of the proof of global existance of solutions to (21).

5.3.2 Proof of Global Existence

We now return to (21).

Proof. of global existence. Define $u_{-1} \equiv 0$ and u_j solves (21) with $\Box u_j = u_{j-1}^p$. For the sake of brevity we will only prove

$$\||x|^{-\alpha}\beta Z^{\leq 2}u_j\|_{L^pL^pL^2} + \|\partial^{\leq 2}\partial u_j\|_{L^\infty L^2} + \|\partial^{\leq 2}u_j\|_{L^2L^2}(|x| \leq 5) \leq 2C_1\epsilon$$

By the above estimate, our goal reduces to showing

$$\||x|^{-p\alpha}\beta^p Z^{\leq 2}u_{j-1}^p\|_{L^1L^1L^2} + \|\partial^{\leq 2}u_{j-1}^p\|_{L^1L^2} \leq 2C_1\epsilon$$

Since $|Z^{\leq 2}u^p| \lesssim |u|^{p-1}|Z^{\leq 2}u| + |u|^{p-2}|Z^{\leq 1}u|^2$, we have

$$\begin{aligned}\||x|^{-p\alpha}\beta^p Z^{\leq 2}u_{j-1}^p\|_{L^1L^1L^2} &\lesssim \||x|^{-\alpha}\beta u_{j-1}\|_{L^pL^pL^\infty}^{p-1}\||x|^{-\alpha}\beta Z^{\leq 2}u_{j-1}\|_{L^pL^pL^2} \\ &\quad + \||x|^{-\alpha}\beta u_{j-1}\|_{L^pL^pL^\infty}^{p-2}\||x|^{-\alpha}\beta u_{j-1}\|_{L^pL^pL^\infty}\||x|^{-\alpha}\beta Z^{\leq 1}u\|_{L^pL^pL^4}^2 \\ &\lesssim \||x|^{-\alpha}\beta Z^{\leq 2}u\|_{L^pL^pL^2}^p \\ &= (LHS)^p\end{aligned}$$

where the second line follows from Sobolev inequalities on $\mathbb{S}^2$.
It remains to show $\|\partial^{\leq 2}u_{j-1}^p\|_{L^1L^2} \leq C_1\epsilon$. We have

$$|\partial^{\leq 2}u_{j-1}^p| = \mathcal{O}\left(|u_{j-1}|^{p-1}|\partial^{\leq 2}u_{j-1}| + |u_{j-1}|^{p-2}|\partial^{\leq 1}u_{j-1}|^2\right)$$

Case 1: $\|\partial^{\leq 2}u_{j-1}^p\|_{L^1L^2(|x|<5)}$

We first state the following proposition:

$$0 < p < q < r \leq \infty \Rightarrow L^p \cap L^r \subseteq L^q$$

and we have

$$\|f\|_{L^q} \leq \|f\|_{L^p}^\lambda \|f\|_{L^r}^{1-\lambda}$$

where $\frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r}$.

We will also use the following estimate:

$$\begin{aligned}\|f\|_{L^\infty(|x|\leq 5)} &\lesssim \|\partial^{\leq 1}f\|_{L^4(|x|\leq 6)} \\ &\lesssim \|\partial^{\leq 1}f\|_{L^6(\mathbb{R}^3)}^{1-\frac{1}{p-1}}\|\partial^{\leq 1}f\|_{L^{12/(p-1)}(|x|\leq 6)}^{\frac{1}{p-1}} \\ &\lesssim \|\partial^{\leq 1}\partial f\|_{L^2}^{1-\frac{1}{p-1}}\|\partial^{\leq 2}f\|_{L^2}^{\frac{1}{p-1}}\end{aligned}$$

Now we evaluate the desired bounds

$$\begin{aligned}\|u_{j-1}^{p-1}\partial^{\leq 2}u_{j-1}\|_{L^1L^2(|x|\leq 5)} &\lesssim \|u_{j-1}\|_{L^{2(p-1)}L^\infty(|x|\leq 5)}^{p-1}\|\partial^{\leq 2}u_{j-1}\|_{L^2L^2(|x|\leq 5)} \\ &\lesssim \|\partial^{\leq 2}u_{j-1}\|_{L^2L^2(|x|<6)}\|\partial^{\leq 1}\partial u_{j-1}\|_{L^\infty L^2}^{p-2}\|\partial^{\leq 2}u_{j-1}\|_{L^2L^2(|x|\leq 5)} \\ &\lesssim (LHS_{j-1})^p\end{aligned}$$

$$\begin{aligned}\|u_{j-1}^{p-2}(\partial^{\leq 1}u_{j-1})^2\|_{L^1L^2(|x|\leq 5)} &\lesssim \|u_{j-1}\|_{L^\infty L^\infty}^{p-2}\|\partial^{\leq 1}u_{j-1}\|_{L^2L^4(|x|\leq 5)}^2 \\ &\lesssim \|\partial^{\leq 1}u_{j-1}\|_{L^\infty L^6}^{p-2}\|\partial^{\leq 2}u_{j-1}\|_{L^2L^2(|x|\leq 6)}^2 \\ &\lesssim \|\partial^{\leq 1}\partial u\|_{L^\infty L^2}^{p-2}\|\partial^{\leq 2}u_{j-1}\|_{L^2L^2(|x|\leq 6)}^2 \\ &\lesssim (LHS_{j-1})^p\end{aligned}$$

Case 2: $\|\partial^{\leq 2} u_{j-1}^p\|_{L^1L^2(|x|\geq 5)}$

We begin with a lemma:

Lemma 5.3. *Weighted Sobolev Estimates Let $\beta \in \mathbb{R}$ and $R \geq 3$. Then for $n = 3$*

(1) For $2 \leq p \leq q \leq \infty$ we have

$$\|r^\beta u\|_{L^q_r L^\infty_\omega(|x|\geq R+1)} \lesssim \|r^{\beta - \frac{2}{p} + \frac{2}{q}} Z^{\leq 2} u\|_{L^p_r L^2_\omega(|x|\geq R)}$$

(2) For $2 \leq p \leq q \leq 4$ we have

$$\|r^\beta u\|_{L^q_r L^4_\omega(|x|\geq R+1)} \lesssim \|r^{\beta - \frac{2}{p} + \frac{2}{q}} Z^{\leq 1} u\|_{L^p_r L^2_\omega(|x|\geq R)}$$

Proof. Sobolev embeddings on $\mathbb{R} \times \mathbb{S}^2$ give

$$\|v\|_{L^\infty_r L^\infty_\omega([j,j+1]\times S^2)} \lesssim \left(\int_{j-1}^{j+2} \int |Z^{\leq 2} v|^2 \, d\omega dr\right)^{1/2}$$

This implies

$$\|r^\beta v\|_{L^\infty_r L^\infty_\omega([j,j+1]\times S^2)} \lesssim \|r^{\beta-1} Z^{\leq 2} u\|_{L^2_r L^2_\omega(|x|\in[j-1,j+2])}$$

We use this in calculating

$$\begin{aligned}
\|r^\beta v\|_{L^q L^\infty}(|x| \in [j, j+1]) &\lesssim \|r^{\beta + \frac{2}{q}} v\|_{L^\infty L^\infty(|x|\in[j,j+1])} \\
&\lesssim \|r^{\beta + \frac{2}{q} - 1} Z^{\leq 2} v\|_{L^2 L^2(|x|\in[j-1,j+2])} \\
&\lesssim \|r^{\beta + \frac{2}{q} - 1 + (\frac{1}{2} - \frac{1}{p})2} Z^{\leq 2} v\|_{L^p L^2(|x|\in[j-1,j+2])}
\end{aligned}$$

where the last line follows using Hölder's inequality. ☐

We now return to our calculation.

$$\begin{aligned}
u_{j-1}^{p-1}\partial^{\leq 2} u_{j-1}\|_{L^1 L^2_r L^2_\omega(r\geq 5)} &\lesssim \|r^{\frac{\alpha}{p-1}} u_{j-1}\|_{L^p L_r^{\frac{2p(p-1)}{p-2}} L^\infty_\omega(r\geq 5)} \|r^{-\alpha}(1-\beta)\partial^{\leq 2} u_{j-1}\|_{L^p L^p L^2} \\
&\lesssim \|r^{-\alpha} Z^{\leq 2} u_{j-1}\|^{p-1}_{L^p L^p L^2(r\geq 4)} \|r^{-\alpha}(1-\beta) Z^{\leq 2} u_{j-1}\|_{L^p L^p L^2} \\
&\lesssim (LHS_{j-1})^p
\end{aligned}$$

$$\begin{aligned}
\|u_{j-1}^{p-2}(\partial^{\leq 1} u_{j-1})^2\|_{L^1 L^2 L^2(r\geq 5)} &\lesssim \|r^{\frac{2}{p-2}\left(\alpha - \frac{2}{p} + \frac{2}{4}\right)} u_{j-1}\|^{p-2}_{L^p L^\infty L^\infty(r\geq 5)} \|r^{-\alpha + \frac{2}{p} - \frac{2}{4}} \partial^{\leq 1} u_{j-1}\|^2_{L^p L^4 L^4(r\geq 5)} \\
&\lesssim \|r^{-\alpha} Z^{\leq 2} u_{j-1}\|^{p-2}_{L^p L^p L^2(r\geq 4)} \|r^{-\alpha} Z^{\leq 2} u_{j-1}\|^2_{L^p L^p L^2(r\geq 4)} \\
&\lesssim (LHS_{j-1})^p
\end{aligned}$$

Now that we have established boundedness, showing the sequence is Cauchy follows as before. ☐

6 The Wave Equation on the Schwarzschild Spacetime

6.1 Introduction

The Schwarzschild spacetime arises in General Relativity and models a solution to Einstein's vacuum equations in the presence of a stationary black hole. The metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\sigma_s^2$$

where $r > 2M$. We can write this in matrix form as

$$(g_{\alpha\beta}) = \begin{bmatrix} -\left(1 - \frac{2M}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2M}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{bmatrix}$$

For the inverse of this matrix we write $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$. The wave equation on this geometric background is given by

$$\begin{aligned} \Box_g u &= |g|^{-1/2}\partial_\alpha\left(g^{\alpha\beta}|g|^{1/2}\partial_\beta u\right) \\ &= -\left(1 - \frac{2M}{r}\right)^{-1}\partial_t^2 u + r^{-2}\partial_r\left\{\left(1 - \frac{2M}{r}\right) r^2 \partial_r u\right\} + \nabla\!\!\!/ \cdot \nabla\!\!\!/ u \end{aligned}$$

As an aside we note that $r = 2M$ is a null hypersurface, so we are not in the same setting as the exterior obstacle problem.
As before we have a conserved energy quantity. If $\Box_g u = 0$, the energy is given by

$$E[u](t) = \frac{1}{2}\int_{r>2M}\left[(1 - \frac{2M}{r})^{-1}(\partial_t u)^2 + (1 - \frac{2M}{r})(\partial_r u)^2 + |\nabla\!\!\!/ u|^2\right] r^2 \; dr d\omega$$

To see this is a conserved quantity, we calculate

$$\begin{aligned} 0 &= \int_0^t \int_{r>2M} \Box_g u \partial_t u r^2 \; dr d\omega dt \\ &= \int_0^t \int \left(-(1 - \frac{2M}{r})^{-1}\partial_t^2 u + r^{-2}\partial_r\left\{\left(1 - \frac{2M}{r}\right) r^2\partial_r u\right\} + \nabla\!\!\!/ \cdot \nabla\!\!\!/ u\right)\partial_t u r^2 \; dr d\omega dt \\ &= \int_0^t \int \left[-\frac{1}{2}\partial_t[(1 - \frac{2M}{r})^{-1}(\partial_t u)^2] - \frac{1}{2}\partial_t[(1 - \frac{2M}{r})(\partial_r u)^2] - \frac{1}{2}\partial_t|\nabla\!\!\!/ u|^2\right] r^2 \; dr d\omega dt \end{aligned}$$

The desired result then follows using the fundamental theorem of calculus.

6.2 Trapping

A new phenomenon that occurs in the Schwarzschild spacetime is that of trapping. Trapping occurs when a null geodesic remains within a compact set for all time. In the case of Schwarzschild, this occurs at $r = 3M$, which is referred to as the photon sphere.

We define affine parameters for geodesics:

$$\frac{1}{2}\left[-(1-\frac{2M}{r})^{-1}\tau^2+(1-\frac{2M}{r})\xi^2+r^{-2}\Theta^2+r^{-2}\sin^{-2}\theta\Phi^2\right]$$

satisfying

$$\dot{t}=-(1-\frac{2M}{r})^{-1}\tau \quad \dot{\tau}=0 \qquad \dot{r}=(1-\frac{2M}{r})\xi \quad \dot{\xi}=0$$
$$\dot{\phi}=r^{-2}\sin^{-2}\theta\Theta^2 \quad \dot{\Theta}=0 \qquad \dot{\theta}=r^{-2}\Theta \quad \dot{\Theta}=r^{-2}\sin^{-3}\theta\cos\theta$$

We have the conserved quantities

$$(1-\frac{2M}{r})\dot{t}=E \qquad r^2\sin^2\theta\dot{\phi}=L$$

WLOG take $\theta\equiv\frac{\pi}{2}$. Since we are concerned with null geodesics, we have $ds=0$. This implies

$$\begin{aligned}0&=-(1-\frac{2M}{r})\dot{t}^2+(1-\frac{2M}{r})^{-1}\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2\\&=-(1-\frac{2M}{r})^{-1}E^2+(1-\frac{2M}{r})^{-1}\dot{r}^2+\frac{1}{r^2}L^2\\\dot{r}^2&=E^2-(1-\frac{2M}{r})r^{-2}L^2=:V(r)\end{aligned}$$

In general, there are 3 possible cases

(1) V hs no real roots $\Rightarrow r$ monotone $\Rightarrow$ no trapping

(2) V has 2 real roots $\Rightarrow$ no trapping

(3) V has a repeated real root. I

n our equation we have $V'(r_0)=0\Rightarrow r_0=3M$. Thus we find trapping at the photon sphere, $r=3M$.

6.3 Local Energy

We next use the now familiar multiplier method to obtain local energy estimates.

$$\begin{aligned}&-\int\Box_g u(1-\frac{2M}{r})f(r)\partial_r ur^2\ drd\sigma(\omega)dt\\&\quad=\int\left[(1-\frac{2M}{r})^{-1}\partial_t^2u-r^{-2}\partial_r(1-\frac{2M}{r})r^2\partial_r u-\not\nabla\cdot\not\nabla u\right](1-\frac{2M}{r})f(r)\partial_r ur^2\ drd\sigma dt\\&\quad=\int\partial_t uf(r)\partial_r ur^2\ drd\omega\Big|_0^t-\int\frac{1}{2}f(r)\partial_r(\partial_t u)^2r^2\ drd\omega dt+\int(1-\frac{2M}{r})\partial_r u\partial_r\left[(1-\frac{2M}{r})f(r)\partial_r u\right]r^2\ drd\sigma dt\\&\qquad+\int\not\nabla u\cdot(1-\frac{2M}{r})f(r)\not\nabla\partial_r ur^2\ drd\sigma dt\\&\quad=\int f(r)\partial_t u\partial_r ur^2\ drd\omega\Big|_0^t-\frac{1}{2}\int f(r)\partial_r(\partial_t u)^2r^2\ drd\omega dt+\int f'(r)(1-\frac{2M}{r})^2(\partial_r u)^2r^2\ drd\omega dt\\&\qquad+\int\frac{1}{2}f(r)\partial_r\left[(1-\frac{2M}{r})\partial_r u\right]^2r^2\ drd\omega dt+\int\frac{1}{2}(1-\frac{2M}{r})f(r)\partial_r|\not\nabla u|^2r^2\ drd\omega dt\end{aligned}$$

We pause our calculations to note that the Lagrangian, which we wish to introduce in the equation, is given by

$$-(1-\frac{2M}{r})^{-1}(\partial_t u)^2+(1-\frac{2M}{r})(\partial_r u)^2+|\nabla\!\!\!/|^2=:L$$

Continuing the calculations above:

$$-\int \Box_g u(1-\frac{2M}{r})f(r)\partial_r ur^2\ drd\sigma(\omega)dt$$
$$=\int f(r)\partial_t u\partial_r ur^2\ drd\omega\Big|_0^t+\int\frac{1}{2}f(r)\partial_r\left[-(\partial_t u)^2+(1-\frac{2M}{r})^2(\partial_r u)^2+(1-\frac{2M}{r})|\nabla\!\!\!/ u|^2\right]r^2\ drd\omega dt$$
$$-\int\frac{1}{2}f(r)\left[\partial_r(1-\frac{2M}{r})\right]|\nabla\!\!\!/ u|^2r^2\ drd\omega dt+\int\left[f'(r)(1-\frac{2M}{r})^2(\partial_r u)^2+(1-\frac{2M}{r})\frac{f(r)}{r}|\nabla\!\!\!/ u|^2\right]r^2\ drd\omega dt$$
$$=\int f(r)\partial_t u\partial_r ur^2\ drd\omega\Big|_0^t-\frac{1}{2}\int r^{-2}\partial_r(r^2f(r))(1-\frac{2M}{r})\left[-(1-\frac{2M}{r})^{-1}(\partial_t u)^2+(1-\frac{2M}{r})(\partial_r u)^2+|\nabla\!\!\!/|^2\right]\ drd\omega dt$$
$$\int\left[f'(r)(1-\frac{2M}{r})^2(\partial_r u)^2+(1-\frac{3M}{r})\frac{f(r)}{r}|\nabla\!\!\!/ u|^2\right]r^2\ drd\omega dt$$

Next we use another multiplier and find

$$-\int\Box_g u\left[\frac{1}{2}(1-\frac{2M}{r})r^{-2}\partial_r(r^2f(r))\right]ur^2\ drd\omega dt$$
$$=\int\frac{1}{2}\left[(1-\frac{2M}{r})r^{-2}\partial_r(r^2f(r))\right](\partial_t u)ur^2\ drd\omega\Big|_0^t+\frac{1}{2}\int\left[(1-\frac{2M}{r})r^{-2}\partial_r(r^2f(r))\right](L)r^2\ drd\omega dt$$
$$-\frac{1}{4}\int r^{-2}\partial_r\left[(1-\frac{2M}{r})r^2\partial_r\left((1-\frac{2M}{r})r^{-2}\partial_r(r^2f(r))\right)\right]u^2r^2\ drd\omega dt$$

We define

$$\ell(f)=-\frac{1}{4}r^{-2}\partial_r\left[(1-\frac{2M}{r})r^2\partial_r\left((1-\frac{2M}{r})r^{-2}\partial_r(r^2f(r))\right)\right]$$

Combining the above calculations, we obtain

$$-\int\Box_g u\left\{(1-\frac{2M}{r})f(r)\partial_r u+\frac{1}{2}(1-\frac{2M}{r})r^{-2}\partial_r(r^2f(r))u\right\}r^2\ drd\omega dt$$
$$=\int f(r)\partial_t u\partial_r ur^2\ drd\omega\Big|_0^t+\int\left[r^{-2}(1-\frac{2M}{r})\partial_r(r^2f)\right]u\partial_t ur^2\ drd\omega\Big|_0^t+\int f'(r)(1-\frac{2M}{r})^2(\partial_r u)^2r^2\ drd\omega dt$$
$$+\int\frac{f(r)}{r}(1-\frac{3M}{r})|\nabla\!\!\!/ u|^2r^2\ drd\omega dt+\int\ell(f)u^2r^2\ drd\omega dt$$

As before, we have a "wish list" for f:

$$\cdot f\in C^2\quad\cdot f\text{ bdd}\quad\cdot f'\lesssim\frac{1}{r}\quad\cdot f'>0\quad\cdot f=\begin{cases}<0 & r<3M\\ >0 & r>3M\end{cases}\quad\cdot\ell(f)>0$$

Unfortunately, it turns out that the 4th and last conditions are incompatible. To begin, let's take

$$f(r)=r^{-2}\left((r-3M)(r+2M)+6M^2\log(\frac{2-2M}{M})\right)$$

Using, e.g. Mathematica, we find

$$f'(r) = \frac{M}{r^3(r-2M)}\left[r^2 + 16Mr - 24M^2 - 12(r-2M)\log(\frac{r-2M}{M})\right] \gtrsim \frac{1}{r(r-2M)}$$

and

$$\ell(f) = \frac{M(7r^2 - 44Mr + 72M^2)}{2r^6} \gtrsim \frac{1}{r^4}$$

We need to smooth out the log blow up. We set

$$a(x) = \begin{cases} -\frac{1}{\epsilon}\frac{\epsilon x+1}{(\epsilon x+1)-1} & x \le \frac{1}{\epsilon} \\ x & x > \frac{1}{\epsilon} \end{cases}$$

Then $a \in C^2(\mathbb{R} \setminus \{-\frac{1}{\epsilon}\})$ with a jump in the 2nd derivative at $-\frac{1}{\epsilon}$.
We replace our first function f with

$$f(r) = r^{-2}\left((r-3M)(r+2M) + 6M^2 a\left(\log(\frac{2-2M}{M})\right)\right)$$

and set $r_{-\frac{1}{\epsilon}}$ to solve $\log(\frac{r_{-\frac{1}{\epsilon}}-2M}{M}) = -\frac{1}{\epsilon}$.
Let's revisit $-\int \Box_g u\left[\frac{1}{2}(1-\frac{2M}{r})r^{-2}\partial_r(r^2 f(r))\right] ur^2\, dr d\omega dt$. There is an alternate expression given by

$$\begin{aligned} &-\int \Box_g u\left[\frac{1}{2}(1-\frac{2M}{r})r^{-2}\partial_r(r^2 f(r))\right] ur^2\, dr d\omega dt \\ &\quad = \int \frac{1}{2}\left[r^{-2}\partial_r(r^2 f(r))\right](\partial_t u)ur^2\, dr d\omega\Big|_0^t + \frac{1}{2}\int\left[(1-\frac{2M}{r})r^{-2}\partial_r(r^2 f(r))\right](L)r^2\, dr d\omega dt \\ &\quad + \int \frac{1}{2}\left[(1-\frac{2M}{r})\partial_r\left(r^{-2}(1-\frac{2M}{r})\partial_r(r^2 f(r))\right)\right] u\partial_r u r^2\, dr d\omega dt \end{aligned}$$

We denote the 3rd term in this expression by $\circledast$ and note

$$\circledast = \frac{1}{4}\int (1-\frac{2M}{r_{-\frac{1}{\epsilon}}})^2[f''(r^-_{-\frac{1}{\epsilon}}) - f''(r^+_{-\frac{1}{\epsilon}})]u^2 r^2_{-\frac{1}{\epsilon}}\, d\omega dt + \int \ell(f)u^2 r^2\, dr d\omega dt$$

We calculate

$$f''(r^-_{-\frac{1}{\epsilon}}) - f''(r^+_{-\frac{1}{\epsilon}}) = \frac{6M}{r^2_{-\frac{1}{\epsilon}}}a''(-\frac{1}{\epsilon}^-)(\frac{1}{r_{-\frac{1}{\epsilon}}-2M})^2$$

$$a'' = \frac{-2\delta\epsilon}{\delta - 1 + \delta\epsilon x)^3} \Rightarrow a''|_{-\frac{1}{\epsilon}} = 2\delta\epsilon > 0$$

For $2M \le r \le r_{-\frac{1}{\epsilon}}$ we have that $a(x) < 0$ for $x < -\frac{1}{\epsilon}$ so $f < 0$ here as we wished. And we have

$$f'(r) = \frac{M(r+12M)}{r^3} - 12\frac{M^2}{r^3}a\left(\log(\frac{r-2M}{M})\right) + \frac{6M^2}{r^2}\frac{1}{r-2M}a'\left(\log(\frac{r-2M}{M})\right) \gtrsim \frac{1}{r(r-2M)}$$

since $a'(x) = \frac{1}{\delta\epsilon x+\delta-1)^2}$.
Next we consider $\ell(f)$ and find on $[2M, r_{-\frac{1}{\epsilon}}]$

$$\begin{aligned}\ell(f) &= \ell(r^{-2}(r-3M)(r+2M)) + \ell\left(\frac{6M^2}{r^2}a\left(\log(\frac{r-2M}{M})\right)\right)\\ &= \frac{M(7r^2-26Mr+18M^2)}{2r^6} - \frac{9M^2(r-3M)}{r^6}a'\left(\log(\frac{r-2M}{M})\right) + \frac{15M^2}{2}\frac{1}{r^5}a''\left(\log(\frac{r-2M}{M})\right)\\ &\quad - \frac{3M^2}{2}\frac{1}{r^4}\frac{1}{r-2M}a'''\left(\log(\frac{r-2M}{M})\right)\end{aligned}$$

The 4th term is the most problematic. To deal with this, we integrate against u^2r^2 and find

$$\begin{aligned}&\frac{1}{4}\int_{2M}^{r_{-\frac{1}{\epsilon}}} 6M^2\frac{1}{r^4}\frac{1}{r-2M}a'''\left(\log(\frac{r-2M}{M})\right)u^2r^2\ dr\\ &\quad= \frac{1}{4}\int_{2M}^{r_{-\frac{1}{\epsilon}}} 6M^2\frac{1}{r^2}\left[\partial_r a''\left(\log(\frac{r-2M}{M})\right)\right]u^2\ dr\\ &\quad= \frac{1}{4}6M^2\frac{1}{r^2_{-\frac{1}{\epsilon}}}a''(-\frac{1}{\epsilon})u^2(r_{-\frac{1}{\epsilon}}) - \frac{1}{4}6M^2\int\frac{-2}{r^3}a''\left(\log(\frac{r-2M}{M})\right)u^2\ dr\\ &\qquad - \frac{1}{4}6M^2\int\frac{1}{r^2}a''\left(\log(\frac{r-2M}{M})\right)2u\partial_r u\ dr\end{aligned}$$

Next we calculate

$$\begin{aligned}&2\cdot 6M^2\int\frac{1}{r^2}a''\left(\log(\frac{r-2M}{M})\right)2u\partial_r u\ dr\\ &\leq 2\cdot 6M^2\left[\left(\int\frac{1}{r^4}\frac{1}{r-2M}\frac{\left(a''\left(\log(\frac{r-2M}{M})\right)\right)^2}{a'\left(\log(\frac{r-2M}{M})\right)}u^2r^2\ dr\right)^{1/2}\left(\int\frac{1}{r^2}a'\left(\log(\frac{r-2M}{M})\right)\frac{1}{r-2M}(1-\frac{2M}{r})^2(\partial_r u)^2r^2\ dr\right)^{1/2}\right]\\ &\leq 6M^2\left[\frac{1}{2}\int_{2M}^{r_{-\frac{1}{\epsilon}}}\frac{1}{r^4}a'''\left(\log(\frac{r-2M}{M})\right)\frac{1}{r-2M}u^2r^2\ dr + \frac{4}{3}\int_{2M}^{r_{-\frac{1}{\epsilon}}}\frac{1}{r^2}a'\left(\log(\frac{r-2M}{M})\right)\frac{1}{r-2M}(1-\frac{2M}{r})^2(\partial_r u)^2r^2\ dr\right]\end{aligned}$$

And

$$\begin{aligned}&\frac{13}{6}\frac{1}{8}\int 6M^2\frac{1}{r^4}a'''\left(\log(\frac{r-2M}{M})\right)\frac{1}{r-2M}u^2r^2\ dr\\ &\quad\leq \frac{13}{6}\frac{1}{4}6M^2\frac{1}{r^2_{-\frac{1}{\epsilon}}}a''(-\frac{1}{\epsilon})u^2(r_{-\frac{1}{\epsilon}}) + \frac{13}{6}\frac{1}{2}6M^2\int\frac{1}{r^5}a''\left(\log(\frac{r-2M}{M})\right)u^2r^2\ dr\\ &\qquad + \frac{13}{6}\frac{1}{3}6M^2\int\frac{1}{r^2}a'\left(\log(\frac{r-2M}{M})\right)\frac{1}{r-2M}(1-\frac{2M}{r})^2(\partial_r u)^2r^2\ dr\end{aligned}$$

We note that $\frac{13}{6}\frac{1}{8} = \frac{1}{4} + \frac{1}{48}$ and find

$$\int \ell(f)u^2r^2\ dr$$

$$\geq \int \frac{M}{2r^6}(7r^2 - 26Mr + 18M^2)u^2r^2\ dr - \int \frac{9M^2}{r^6}(r-3M)a'\left(\log(\frac{r-2M}{M})\right)u^2r^2\ dr$$

$$+ M^2\int \frac{1}{r^5}a''\left(\log(\frac{r-2M}{M})\right)u^2r^2\ dr + \frac{1}{48}\int 6M^2\frac{1}{r^4}a'''\left(\log(\frac{r-2M}{M})\right)\frac{1}{r-2M}u^2r^2\ dr$$

$$- \int \frac{13}{24}6M^2\frac{1}{r^2_{-\frac{1}{\epsilon}}}a''\left(-\frac{1}{\epsilon}\right)u^2(r_{-\frac{1}{\epsilon}})\ dr - \frac{13}{18}\int 6M^2\frac{1}{r^2}a'\left(\log(\frac{r-2M}{M})\right)\frac{1}{r-2M}(1-\frac{2M}{r})^2(\partial_r u)^2r^2\ dr$$

The 1st and 4th terms go to 0 as $\epsilon \to 0$. The 2nd and 3rd terms are nonnegative. The 6th term is a fraction of another term from earlier. Thus it is only the 5th term that presents a problem.
Let's pause for a quick recap. We now have the following expression:

$$-\int \Box_g u\left\{(1-\frac{2M}{r})f(r)\partial_r u + \frac{1}{2}(1-\frac{2M}{r})r^{-2}\partial_r(r^2f(r))u\right\}r^2\ drd\omega dt$$

$$= \int f(r)\partial_t u\partial_r ur^2\ drd\omega\Big|_0^t + \int \left[r^{-2}\partial_r(r^2f)\right]u\partial_t ur^2\ drd\omega\Big|_0^t + \int f'(r)(1-\frac{2M}{r})^2(\partial_r u)^2r^2\ drd\omega dt$$

$$+ \int \frac{f(r)}{r}(1-\frac{3M}{r})|\nabla\!\!\!/ u|^2r^2\ drd\omega dt + \int \ell(f)u^2r^2\ drd\omega dt$$

$$+ \frac{1}{4}\int (1-\frac{2M}{r_{-\frac{1}{\epsilon}}})^2[f''(r^-_{-\frac{1}{\epsilon}}) - f''(r^+_{-\frac{1}{\epsilon}})]u^2r^2_{-\frac{1}{\epsilon}}\ d\omega dt$$

We have that the 6th term is nonnegative. For the 6th term it is left to deal with

$$-\frac{7}{24}6M^2\frac{1}{r_{-\frac{1}{\epsilon}}}a''(-\frac{1}{\epsilon}^-)\int u^2(t, r_{-\frac{1}{\epsilon}}\omega)\ dtd\omega$$

Set $\phi^2(r) := \int u^2(t, r\omega)\ dtd\omega$ and fix β such that

$$\beta = \begin{cases} 1 & r < r_{-1} \\ 0 & r > r_0 = 3M \end{cases}$$

Then we have

$$\phi(r) = -\int_r^{3M} \partial_s(\beta\phi)\ ds = -\int_r^{3M} \beta'\phi + \beta\partial_r\phi\ ds$$

Writing $\beta = \frac{\beta^{1/2}}{(s-2M)^{1/2}} \cdot \beta^{1/2}(s-2M)^{1/2}$ we find

$$\phi^2(r) \lesssim \int_r^{3M} |\beta'|\phi^2\ ds - \log(\frac{r-2M}{M})\int_r^{3M}(s-2M)\beta(\partial_r\phi)^2\ ds$$

Thus

$$\phi^2(r_{-\frac{1}{\epsilon}}) \lesssim \int_{r_{-\frac{1}{\epsilon}}}^{3M} |\beta'|\phi^2\ ds + \frac{1}{\epsilon}\int_{r_{-\frac{1}{\epsilon}}}^{3M}(r-2M)\beta(\partial_r\phi)^2\ dr$$

and we have

$$\epsilon\phi^2(r_{-\frac{1}{\epsilon}}) \lesssim \int_{[r_{-\frac{1}{\epsilon}},3M]} \left[(1-\frac{2M}{r})^2 f'(r)(\partial_r\phi)^2 + \ell(r)\phi^2\right] r^2\ dr$$

Since $a''(-\frac{1}{\epsilon})(\frac{1}{r_{-\frac{1}{\epsilon}}-2M})^2 = c\delta\cdot\epsilon e^{2/\epsilon}$..., if we choose $\delta << e^{-2/\epsilon}$ we get

$$-\frac{7}{24}6M^2\int\frac{1}{r^2_{-\frac{1}{\epsilon}}}a''\left(-\frac{1}{\epsilon}^{-}\right)u^2(t,r_{-\frac{1}{\epsilon}}\omega)\ dtd\omega \geq -\frac{1}{2}\int\left[f'(r)(1-\frac{2M}{r})^2(\partial_r u)^2 + \ell(r)u^2\right] r^2\ drd\omega dt$$

We note that on $[2M, r_{-\frac{1}{\epsilon}}]$,

$$f' = \frac{M(r+2M)}{r^3} - \frac{12M^2}{r^3}a\left(\log(\frac{r-2M}{M})\right) + \frac{6M^2}{r^2}a'\left(\log(\frac{r-2M}{M})\right)\frac{1}{r-2M} \geq \frac{6M^2}{r^2}a'\left(\log(\frac{r-2M}{M})\right)\frac{1}{r-2M}$$

Combining our results we find

$$-\int\Box_g u\left\{(1-\frac{2M}{r})f(r)\partial_r u + \frac{1}{2}(1-\frac{2M}{r})r^{-2}\partial_r(r^2 f(r))u\right\} r^2\ drd\omega dt$$
$$\geq \int f(r)\partial_t u\partial_r u r^2\ drd\omega\Big|_0^t + \int\left[r^{-2}\partial_r(r^2 f)\right]u\partial_t u r^2\ drd\omega\Big|_0^t + \int\left[\frac{(1-\frac{2M}{r})}{r^2\left(1-\log(\frac{r-2M}{r})\right)^2}(\partial_r u)^2 + \frac{(1-\frac{3M}{r})^2}{r}|\nabla u|^2 \ldots\right.$$

7 Quasilinear Wave Equations

In this section we will consider the quasilinear wave equation given by

$$\begin{cases} g^{\alpha\beta}(u,u')\partial_\alpha\partial_\beta u = F(u,u') \\ u(0,\cdot) = \epsilon f; \quad \partial_t u(0,\cdot) = \epsilon g \end{cases}$$

with $(t,x)\in\mathbb{R}\times\mathbb{R}^n$ and where $f,g\in C_c^\infty$, $g^{\alpha\beta}(0)\partial_\alpha\partial_\beta = \Box$, and $F(0) = dF(0) = 0$ (i.e. $F(u') = O(|u'|^2)$.

We begin with a heuristic discussion of what we may expect to happen in different dimensions. For these purposes we restrict to the case $g^{\alpha\beta}\partial_\alpha\partial_\beta = \Box$ and F is quadratic. Then from local energy we have

$$\|u'(t,\cdot)\|_{L^2} \lesssim \epsilon + \int\|\Box u(s,\cdot)\|_{L^2}\ ds$$

and $\Box u$ can be written as the sum of the product of (at most) two components of the gradient of u. The key to long time estimates is the integrability of $\|\partial_\alpha u\partial_\beta u\|_{L^2}$. This ultimately depends on the integrability of $(1+t)^{-\frac{n-1}{2}}$. We have

$$\int(1+t)^{-\frac{n-1}{2}} = \begin{cases} <\infty & n\geq 4 \\ \log t & n = 3 \\ \infty & n\leq 2 \end{cases}$$

Thus we anticipate global existence for $n \geq 4$, almost global existence for $n = 3$, and difficulties in obtaining existence for $n \leq 2$. We will show that we do in fact have global existence for $n \geq 4$ and almost global existence for $n = 3$. In the $n = 3$ case we show there exists a solution on $[0, T_*)$ for $T_* = \exp(c/\epsilon)$. Since T_* increases exponentially to infinity as $\epsilon \to 0$ we say the existence is almost global. We will not prove local existence for $n = 2$ and $n = 1$, but results as in the $n = 3$ case hold for $T_* = \left(\frac{c}{\epsilon}\right)^2$ for $n = 2$ and $T_* = \frac{c}{\epsilon}$ for $n = 1$.

7.1 Local Existence

We consider the equation

$$\begin{cases} g^{\alpha\beta}(u, u')\partial_\alpha\partial_\beta u = F(u, u') \\ u(0, \cdot) = u_0; \quad \partial_t u(0, \cdot) = u_1 \end{cases} \tag{23}$$

with $g, F \in C^\infty$ and all their derivatives are $O(1)$, $F(0, 0) = 0$, and $|g^{\alpha\beta} - m^{\alpha\beta}| < \frac{1}{2}$ where $m^{\alpha\beta}$ are the coefficients of the wave equation on the Minkowski metric (i.e. the flat wave equation).

Theorem 7.1. *If $s > n+2$ and $(u_0, u_1) \in H^{s+1} \times H^s$, then there exists a $T > 0$ such that (22) has a unique solution such that*

$$\sum_{|\alpha| \leq s+1} \|\partial^\alpha u(t, \cdot)\|_{L^2} < \infty \quad 0 \leq t \leq T.$$

Moreover, if T_* is the supremum of all such T, then either $T_* = \infty$ or

$$\sum_{|\alpha| \leq \frac{s+3}{2}} |\partial^\alpha u(t, x)| \notin L^\infty_{t,x}([0, T_*) \times \mathbb{R}^n)$$

Before proving the local existence theorem, we state and prove the following preliminary proposition.

Proposition 7.2. *Suppose $u \in C^2([0, T] \times \mathbb{R}^n)$ vanishes for $|x|$ large and*

$$g^{\alpha\beta}(t, x)\partial_\alpha\partial_\beta u = F$$

$$r^{\alpha\beta}(t, x) = g^{\alpha\beta}(t, x) - m^{\alpha\beta}$$

Then if $\sum_{\alpha\beta} |r^{\alpha\beta}| < \frac{1}{2}$ for $0 \leq t \leq T$ then

$$\|u'(t, \cdot)\|_{L^2} \leq 2 \left(\|u'(0, \cdot)\|_{L^2} + \int_0^t \|F(s, t)\|_{L^2} \, ds \right) \exp \left(\int_0^t 2 \sum_{\alpha,\beta,\gamma} \|\partial_\alpha g^{\beta\gamma}(s, \cdot)\|_{L^\infty} \, ds \right)$$

Proof. We will use the Einstein notation so that the recurrence of parameter indicates summing over all possible values. We will also use the convention that $(t, x) = (x_0, x_1, \ldots, x_n)$, Greek symbols range from 0 to n, and Roman symbols range from 1 to n.
We will leave out details but note they can be obtained by taking $\Box_g = g^{\alpha\beta}\partial_\alpha\partial_\beta$, considering $\int \Box_g u \partial_t u$, and integrating by parts. We begin with

$$\begin{aligned} e_0(u) &:= 2g^{0\alpha}\partial_0 u \partial_\alpha u - g^{\alpha\beta}\partial_\alpha u \partial_\beta u \\ &= |u'|^2 + 2r^{0\alpha}\partial_0 u \partial_\alpha u - r^{\alpha\beta}\partial_\alpha u \partial_\beta u \end{aligned}$$

Define $E(t) = \int e_0(u)(t,x)\ dx$. Then

$$\partial_t E(t) = \int 2\partial_t u \Box_g u\ dx + \int 2\left(\partial_0 g^{0\alpha}\right)\partial_t u \partial_\alpha u - \left(\partial_0 g^{\alpha\beta}\right)\partial_\alpha u \partial_\beta u + 2\left(\partial_k g^{jk}\right)\partial_t u \partial_j u\ dx$$

We define R to be the integrand of the second term above and note that

$$\frac{1}{2}|u'|^2 \leq e_0(u) \leq 2|u'|^2$$

Thus we have

$$|R| \leq 2|u'|^2 \sum |\partial_\gamma g^{\alpha\beta}| \leq 4e_0(u) \sum |\partial_\gamma g^{\alpha\beta}|$$

It follows that

$$\begin{aligned}\partial_t E(t) &\leq 2\|\partial_u u\|_{L^2}\|F\|_{L^2} + 4\sum \|\partial_\gamma g^{\alpha\beta}\|_{L^\infty} E(t) \\ &\leq 4E^{1/2}(t)\|F\|_{L^2} + 4\sum \|\partial_\gamma g^{\alpha\beta}\|_{L^\infty} E(t)\end{aligned}$$

And therefore

$$\partial_t E^{1/2}(t) \leq 2\|F\|_{L^2} + 2\sum \|\partial_\gamma g^{\alpha\beta}\|_{L^\infty}$$

Thus

$$\begin{aligned}\partial_t\left(E^{1/2}(t)\exp\left(-2\int_0^t \sum \|\partial_\gamma g^{\alpha\beta}(s,\cdot)\|_{L^\infty}\ ds\right)\right) &\leq 2\|F\|_{L^2}\exp\left(-2\int_0^t \sum \|\partial_\gamma g^{\alpha\beta}(s\cdot)\|_{L^\infty}\ ds\right) \\ &\leq 2\|F(t,\cdot)\|_{L^2}\end{aligned}$$

Ginally this gives

$$E^{1/2}(t) \lesssim \exp\left(2\int_0^t \sum \|\partial_\gamma g^{\alpha\beta}(s\cdot)\|_{L^\infty}\ ds\right)\left(\|u'(0,\cdot)\|_{L^2} + \int_0^t \|F(s,cdot)\|_{L62}\ ds\right)$$

□

We are now ready to prove Theorem 7.1

Proof. of 7.1 Define $u_{-1} \equiv 0$ and let u_m solve

$$\begin{cases} g^{\alpha\beta}(u_{m-1}, u'_{m-1})\partial_\alpha\partial_\beta u_m = F(u_{m-1}, u'_{m-1}) \\ u_m(0,\cdot) = f; \quad \partial_t u_m(0,\cdot) = g \end{cases}$$

where $f, g \in \mathcal{S}$.
As usual we first establish boundedness, then show the sequence is Cauchy.

Step 1: Boundedness

Let A be such that $A_0(t) \leq A$ and define

$$A_m := \sum_{|\alpha \leq s} (\|\partial^\alpha u_m(t,\cdot)\|_{L^2} + \|\partial^\alpha u'_m(t,\cdot)\|_{L^2})$$

We want to show that $A_m(t) \le A < \infty$ for all $t \in [0, T]$.

The Sobolev inequalities imply

$$|\partial^\beta u_m(t,\cdot)| \lesssim \sum_{|\alpha|+|\beta| \le \lfloor \frac{n+1}{2} \rfloor} \|\partial^\alpha u_m(t,\cdot)\|_{L^2}$$

We have

$$0 \le |\alpha| \le s+1-\lfloor \frac{n+2}{2} \rfloor \Rightarrow |\partial^\alpha u_m(t,\cdot)| \lesssim A_m(t)$$

And energy implies $A_0(t) \le \frac{A}{2}$. Assume $A_{m-1}(t) \le A$ for all $t \in [0,T]$

We note that

$$g^{\alpha\beta}(u_{m-1}, u'_{m-1})\partial_\alpha \partial_\beta \partial^\mu u_m = \partial^\mu F9u_{m-1}, u'_{m-1}) - \left[\partial^\mu, g^{\alpha\beta}(u_{m-1}, u'_{m-1})\right] \partial_\alpha \partial_\beta u_m$$

For $|\mu| \le s$ we have

$$|\partial^\mu F(v, v')| \lesssim \left(1 + \sum_{|\nabla| \le \lfloor \frac{s+2}{2} \rfloor} \|\partial^\nabla v\|_\infty \right)^{s-1} \sum_{|\mu| \le s+1} |\partial^\mu| v|$$

$$| \left[\partial^\mu, g^{\alpha\beta}(v, v')\right] \partial_\alpha \partial_\beta w| \lesssim \left(1 + \sum_{|\alpha| \le \lfloor \frac{s+3}{2} \rfloor} \|\partial^{\alpha v(t,\cdot)\|_\infty} \right)^{s} \sum_{|\mu| \le s+1} |\partial^\mu w|$$
$$+ \sum_{|\mu| \le \lfloor \frac{s+3}{2} \rfloor} \|\partial^\mu w(t,\cdot)\|_\infty \left(1 + \sum_{|\mu| \le \lfloor \frac{s+3}{2} \rfloor} \|\partial^\mu v(t,\cdot)\|_\infty \right)^{s-1} \sum_{|\mu| \le s+1} |\partial^\mu v|$$

Thus we have

$$\|\partial^\mu F9u_{m-1}, u'_{m-1})\|_{L^2} + \| \left[\partial^\mu, g^{\alpha\beta}(u_{m-1}, u'_{m-1})\right] \partial_\alpha \partial_\beta u_m\|_{L^2}$$

if $\lfloor \frac{n+1}{2} \rfloor + \lfloor \frac{s+2}{2} \rfloor \le s+1$, which is true if and only if $s > n+2$.

Next we calculate

$$\begin{aligned} \|u(t,\cdot)\|_{L^2} &\le \|u(0,\cdot)\|_{L^2} + \int_0^t \|\partial_t u(\tau,\cdot)\|_{L^2} \, d\tau \\ &\le \|u(0,\cdot)\|_{L^2} + t \sup_{0 \le \tau \le t} \|u'(\tau,\cdot)\|_{L^2} \end{aligned}$$

This along with energy gives

$$\begin{aligned} &\|\partial^\mu u_m(t,\cdot)\|_{L^2} + \|\partial^\mu u'_m(t,\cdot)\|_{L^2} \\ &\quad \le C_t \left(\|\partial^\mu u_m(0,\cdot)\|_{L^2} + \|\partial^\mu \partial_t u_m(0,\cdot)\|_{L^2} + C_A \int_0^t (A_m(\tau)+1) \, d\tau \right) \exp\left(\int_0^t 2 \sum \|\partial^\gamma g^{\alpha\beta}\|_\infty \, ds \right) \end{aligned}$$

In other words,

$$A_m(t) \le Ce^{CAt}\left(A_m(0) + C_A \int_0^t A_m(\tau) + 1 \; d\tau\right)$$

Gronwall's inequality tells us

$$A_m(t) \le \tilde{C}e^{CAt}(A + C_A t)\exp(\tilde{C}C_A e^{CAt})$$

Then if t is small enough we get $A_m \le CA$, as desired.

Step 2: Cauchy

We will show

$$c_m(t) = \|u_m(t,\cdot) - u_{m-1}(t,\cdot)\|_{L^2} + \|u'_m(t,\cdot) - u'_{m-1}(t,\cdot)\|_{L^2} = O(2^{-m})$$

Note that

$$\begin{aligned}
&g^{\alpha\beta}(u_{m-1}, u'_{m-1})\partial_\alpha\partial_\beta(u_m - u_{m-1}) \\
&\qquad = F(u_{m-1}, u'_{m-1}) - F(u_{m-2}, u'_{m-2}) - \left(g^{\alpha\beta}(u_{m-1}, u'_{m-1}) - g^{\alpha\beta}(u_{m-2}, u'_{m-2})\right)\partial_\alpha\partial_\beta u_{m-1} \\
&\qquad = O_A(|u_{m-1} - u_{m-2}| + |u'_{m-1} - u'_{m-2}|)(1 + |\partial^2 u_{m-1})
\end{aligned}$$

Energy gives us

$$\begin{aligned}
c_m(t) &\le C\int_0^t \left(1 + \|\partial^2 u_{m-1}\|_\infty\right) c_{m-1}(\tau) \; d\tau \\
&\le C\int_0^t c_{m-1}(\tau) \; d\tau
\end{aligned}$$

Iterating we find

$$\begin{aligned}
c_m(t) &\le C \iint \cdots \int_{0 \le \tau_1 \le \cdots \le \tau_m \le t} c_0(\tau_1) \; d\tau_1 \cdots d\tau_m \\
&\le \frac{C^m t^m}{m!} \sup_{0 \le t \le T} c_0(t) \\
&\le 2^{-m}
\end{aligned}$$

where the last inequality holds for sufficiently small T.

For uniqueness, a similar argument gives

$$\sum_{|\mu| \le 1} \|\partial^\mu(u - \tilde{u})(t,\cdot)\|_{L^2} \le C\int_0^t \sum_{|\mu|\le 1} \|\partial^\mu(u - \tilde{u})(\tau,\cdot)\|_{L^2} \; d\tau$$

And Gronwall's implies

$$\sum_{|\mu| \le 1} \|\partial^\mu(u - \tilde{u})(t,\cdot)\|_{L^2} = 0$$

so that $u \equiv \tilde{u}$.
Finally we consider the continuation condition. Suppose $T_* < \infty$ and

$$\sum_{|\mu| \leq s+1} \|\|\partial^\mu u(t,\cdot)\|_{L^2} < \infty$$

for all $0 \leq t \leq T < T_*$. If

$$\sup_{(t,x)\int[0,T_*)\times\mathbb{R}^n} \sum_{|\alpha|\leq\frac{s+3}{2}} |\partial^\alpha u(t,x)| \leq A < \infty$$

then it suffices to show

$$\sup_{0\leq t<T_*} \sum_{|\alpha|\leq s+1} \|\partial^{\alpha u(t,\cdot)\|_{L^2}<\infty}$$

For if so, $u \in L^\infty([0,T_*] \times H^{s+1}) \cap C^{0,1}([0,T_*], S^s)$ and thus $u(T_*,\cdot)$ and $\partial_t u(T_*),\cdot)$ can be intial values to rerun the local existence argument.
Argue as above...

$$A(t) = \sum_{|\alpha|\leq s+1} \|\partial^\alpha u(t,\cdot)\|_{L^2}$$

Then

$$A(t) \leq C_{T_*,A}\left(A(0) + C_A\int_0^t A(\tau) + 1\ d\tau\right)$$

and Gronwall's inequality finishes the argument. $\square$

7.2 Long Time Existence

We now turn our attention to the question of long time existence. We are concerned with the equation

$$\begin{cases} g^{\alpha\beta}(u')\partial_\alpha\partial_\beta u = F(u') \\ u(0,\cdot) = \epsilon f; \quad \partial_t u(0,\cdot) = \epsilon g \\ f, g \in C_c^\infty \\ F, g^{\alpha\beta} \in C^\infty \\ g^{\alpha\beta}(0)\partial_\alpha\partial_\beta = \Box; \quad F(0) = dF(0) = 0\ (i.e. F(u') = O(|u'|^2)) \end{cases} \tag{24}$$

for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$.
In order to prove long time existence, we will first need to introduce some vector field methods due to Klainerman.

7.2.1 An Introduction to Vector Field Methods

We have the invariant vector fields

$$\partial_\alpha \quad 0 \leq \alpha \leq n$$

$$\Omega_{ij} = x_i\partial_j - x_j\partial_i \quad 1 \leq i < j \leq n$$

$$S = t\partial_t + r\partial_r$$

$$\Omega_{0j} = t\partial_j + x_j\partial_t$$

and we denote the set of these vector fields by Γ.
A key piece of the puzzle is the fact that

$$[\Box, \partial] = [\Box, \Omega] = 0 \qquad [\Box, S] = 2\Box$$

The commutator $[\Box, S]$ gives us that if $\Box u = 0$ then $\Box(Su) = 0$.
Another important piece is the commutator relations between the invariant vector fields.

$$[\Gamma_i, \Gamma_j] = \sum c_{ijk}\Gamma_k \qquad [\Gamma_k, \partial_\alpha] = \sum a_{k\alpha\beta}\partial_\beta$$

We also have

$$|\Gamma^{\leq m}| = \sum_{|\mu|\leq M} |\Gamma^{mu}u|$$

The following is an important estimate in our calculations

Theorem 7.3 (Klainerman-Sobolev)**.** *If $u \in C^\infty(\mathbb{R}^{1+n})$ vanishes for large $|x|$ with $t > 0$, then*

$$(1 + t + |x|)^{\frac{n-1}{2}}(1 + |t - |x||)^{1/2}|u(t, x)| \lesssim \|\Gamma^{\leq \lfloor \frac{n+1}{2} \rfloor}u(t, \cdot)\|_{L^2}$$

We give only partial results of this inequality. First we establish the following lemma

Lemma 7.4. *In $\mathbb{R}^{1+n}_+ \setminus \{0\}$,*

$$(t - r)\partial_r = a_0(t, x)S + \sum_{i=1}^{n} a_i(t, x)\Omega_{0i}$$

where the a_j are smooth, homogeneous of degree 0, and

$$|\partial^\alpha a_j(t, x)| \leq C_\alpha(t + |x|)^{-|\alpha|} \qquad \forall \alpha, |x| > \delta t$$

And

$$(t - r)^2 \sum_{i=0^n} |\partial_i u(t, x)|^2 \leq |Su(t, x)|^2 + \sum_{0\leq j<k\leq n} |\Omega_{jk}u(t, x)|^2$$

Proof. We state the identity:

$$(t - r)\partial_r = \frac{1}{t + r}\left(t\sum_{i=1}^{n} \frac{x_i}{|x|}\Omega_{0i} - rS\right)$$

which gives part 1.
For part 2, we claim

$$(t^2 - |x|^2)\partial_j = -\epsilon_j x_j S + \epsilon_j \sum_{i=0}^{n} x_i\Omega_{ij} \quad j = 0, \dots, n$$

If $j = 0$, then

$$\sum_1^n x_i \Omega_{i0} = \sum_1^n x_i^2 \partial_t + t x_i \partial_i = r^2 \partial_t - t^2 \partial_t + t(t\partial_t + r\partial_r) = (r^2 - t^2)\partial_t + S$$

If $j = 1, \dots, n$, then

$$\sum_1^n x_i \Omega_{ij} = t^2 \partial_j + t x_j \partial_t + \sum_{i=1}^n (x_i x_j \partial_i - x_i^2 \partial_j) = (t^2 + |x|^2)\partial_j + x_j S$$

□

Note that if we fix $\delta > 0$, then Sobolev inequalities give

$$|f(x)|^2 \leq C_{n,\delta} \int_{|y|<\delta} \left| \left(\partial^{\leq \lfloor \frac{n+2}{2} \rfloor} f \right)(x+y) \right|^2 \, dy$$

And on $\mathbb{R} \times \mathbb{S}^{n-1}$,

$$|v(r,\omega)|^2 \leq C_{n,\delta} \sum_{j+|\alpha| \leq \lfloor \frac{n+2}{2} \rfloor} \iint_{|q|<\delta} |\partial_q^j \partial_\omega^\alpha v(r+q,\eta)|^2 \, dq d\sigma(\eta)$$

Proof. of partial Klainerman-Sobolev inequality
WLOG $t + |x| > 1$

Case 1: $|x| \notin [\frac{t}{2}, \frac{3t}{2}]$

This is the better case because we are away from the light cone.

$$\begin{aligned}
(t+|x|)^n |u(t,x)|^2 &\lesssim (t+|x|)^n \sum_{|\alpha| \leq \frac{n+2}{2}} \int_{|y|<\frac{1}{8}} |\partial_y^\alpha (u(t, x + (t+|x|)y))|^2 \, dy \\
&\lesssim \sum_{|\alpha| \leq \frac{n+2}{2}} \int_{|y| \leq \frac{t+|x|}{8}} |[(t+|x|)\partial_y]^\alpha u(t,x+y)|^2 \, dy \\
&\lesssim \left\| \Gamma^{\leq \lfloor \frac{n+2}{2} \rfloor} u(t,\cdot) \right\|_{L^2}^2
\end{aligned}$$

Case 2: $|x| \in [\frac{t}{2}, \frac{3t}{2}]$ We write

$$v(t,q,\omega) = u(t, t + q\omega) = u(t, r\omega)$$

Then

$$\begin{aligned}
t^{n-1}|u(t,x)|^2 &= t^{n-1}|v(t,q,\omega)|^2 \\
&\lesssim C t^{n-1} \sum_{j+|\alpha| \leq \frac{n+2}{2}} \int_{|p|<\frac{1}{2}} \int_{\eta \in S^{n-2}} |\partial_p^j \partial_\omega^\alpha (v(t, q+q-p, \eta))^2 \, dp d\eta \\
&\lesssim C t^{n-1} \sum_{j+|\alpha| \leq \frac{n+2}{2}} \iint_{r \in \left[\frac{t}{2}, \frac{3t}{2}\right]} \left| ((t-r)\partial_r)^j \Gamma^\alpha u(t, r\eta) \right|^2 \, d\eta dr \\
&\lesssim \left\| \Gamma^{\leq \lfloor \frac{n+2}{2} \rfloor} u(t,\cdot) \right\|_{L^2}^2
\end{aligned}$$

□

We will use the facts discussed here to prove global existence for $n \geq 4$ in the next section.

7.2.2 Global Existence for $n \geq 4$

We will prove the following theorem:

Theorem 7.5. *If $n \geq 4$, $f, g \in C_c^\infty$ and $\epsilon > 0$ is sufficiently small, then there exists a global solution to* (23)*.*

Proof. By the proof of the local existence theorem, there exists $T > 0$ such that there is a solution in $[0, T] \times \mathbb{R}^n$. Morever, if $T_* < \infty$, by continuation, if

$$\sup_{[0,T_*)\times\mathbb{R}^n} \left|\partial^{\leq \frac{n+6}{2}} u\right| < \infty \tag{25}$$

then the solution extends past T_*.
Fix $T_* > 0$ and set

$$A(t) = \left\| \left(\Gamma^{\leq s} u\right)'(t, \cdot) \right\|_{L^2(\mathbb{R}^n)}$$

with $s \geq n + 4$. We'll show that

$$\sup_{[0,T_*)} A(t) < \infty \tag{26}$$

if $\epsilon > 0$ sufficiently small.
We claim that (25) $\Rightarrow$ (24). To prove the claim, note that with data supported in $|x| \leq R$, then u is supported in $|x| \leq t + R$. The Klainerman-Sobolev inequality implies

$$(1+t)^{\frac{n-1}{2}} (1 + |t - |x||)^{1/2} \left| \left(\Gamma^{\leq s - \lfloor \frac{n+2}{2} \rfloor} u\right)' \right| \lesssim \left\| \left(\Gamma^{\leq s} u\right)'(t, \cdot) \right\|_{L^2} = A(t) < \infty$$

Applying the fundamental theorem of calculus in the $t - r$ direction, then

$$\left| \Gamma^{\leq s - \lfloor \frac{n+2}{2} \rfloor} u \right| \lesssim \frac{(1 + |t - |x||)^{1/2}}{(1+t)^{\frac{n-1}{2}}} \sup_{[0,T_*)} A(t)$$

which is finite by (25). Since $s > n + 4 \Rightarrow s - \lfloor \frac{n+2}{2} \rfloor \geq \frac{n+6}{2}$, (24) follows. This proves the claim.
Fix A large enough so that $A(0) \leq \frac{A\epsilon}{16}$. Then we'll show

$$A(t) \leq \frac{A\epsilon}{2}$$

for all $0 \leq t \leq T$ with $T < T_*$.
Set

$$E = \{t \in [0, T_*) : A(s) \leq \frac{A\epsilon}{2} \quad \forall 0 \leq s \leq t\}$$

then our goal is to show $E = [0, T_*)$.
By local existence $E \neq \emptyset$. We also have that $A(t)$ is continuous so that E is relatively closed. We will show

$$A(t) \leq A\epsilon \Rightarrow A(t) \leq \frac{A\epsilon}{2}$$

which gives that E is relatively open.

Assume $A(t) \leq A\epsilon$. As above,

$$(1+t)^{\frac{n-1}{2}} \left|\Gamma^{\leq s-\lfloor\frac{n+2}{2}\rfloor} u'(t,x)\right| \lesssim \left\|\Gamma^{\leq s} u'(t,\cdot)\right\|_{L^2} \leq cA\epsilon \tag{27}$$

Now set $r^{\alpha\beta} = m^{\alpha\beta} - g^{\alpha\beta}(u')$. Since $g^{\alpha\beta}(0) = m^{\alpha\beta}$ we have

$$|r^{\alpha\beta}| \leq \frac{C'A\epsilon}{(1+t)^{\frac{n-1}{2}}} \leq \frac{1}{2}$$

for sufficiently small ϵ. And

$$\sum \|\partial_\gamma g^{\alpha\beta}(u')(t,\cdot)\|_{L^\infty} \leq \frac{C'A\epsilon}{(1+t)^{\frac{n-1}{2}}}$$

Since we assume $n \geq 4$ we get integrability:

$$\exp\left(2\int_0^t \sum \|\partial_\gamma g^{\alpha\beta}(u')(s,\cdot)\|_{L^\infty}\ ds\right) \leq \exp\left(2\int_0^\infty \frac{C'A\epsilon}{(1+s)^{\frac{n-1}{2}}}\ ds\right) \leq 2$$

if ϵ is sufficiently small.
Next we will estimate the L^2 norm of $(\Gamma^\mu u)'$ for $|\mu| \leq s$ using energy estimates and the calculation

$$\begin{aligned} g^{\alpha\beta}(u')\partial_\alpha\partial_\beta\Gamma^\mu u &= \Gamma^\mu F(u') + [g^{\alpha\beta}(u')\partial_\alpha\partial_\beta, \Gamma^\mu]u \\ &= \Gamma^\mu F(u') + [\Box, \Gamma^\mu]u - [r^{\alpha\beta}(u')\partial_\alpha\partial_\beta, \Gamma^\mu]u \\ &= \Gamma^\mu F(u') + [\Box, \Gamma^\mu]u - r^{\alpha\beta}(u')[\partial_\alpha\partial_\beta, \Gamma^\mu]u - [r^{\alpha\beta}(u'), \Gamma^\mu]\partial_\alpha\partial_\beta u \end{aligned}$$

We claim tha teach term in the RHS is a linear combination of terms of the form

$$a(u')\Gamma^{\mu_1}u' \cdot \Gamma^{\mu_N}u', \quad |\mu_j| \leq |\mu|$$

where $N \geq 2$ and at most one μ_j satisfies $|\mu_j| > \frac{s+1}{2}$. We will not prove the claim, but refer the reader to Sogge for justification.
Assuming the claim, we can estimate the L^2 norm of the RHS using (26) forall but one term, which we leave with L^2 norm. That is, for $|\mu| \leq s$ we have

$$\begin{aligned} \|g^{\alpha\beta}(u')\partial_\alpha\partial_\beta\Gamma^\mu u(t,\cdot)\|_{L^2} &\leq \frac{C_A\epsilon}{(1+t)^{\frac{n-1}{2}}} \left\|\left(\Gamma^{\leq s}u\right)'(t,\cdot)\right\|_{L^2} \\ &= \frac{C_A\epsilon A(t)}{(1+t)^{\frac{n-1}{2}}} \end{aligned}$$

Using Proposition 7.2 and energy estimates, we find

$$A(t) \leq 4\left(A(0) + \int_0^t \frac{C_A\epsilon A(s)}{(1+s)^{\frac{n-1}{2}}}\ ds\right)$$

Gronwall's inequality then gives

$$\begin{aligned} A(t) &\le 4A(0)\exp\left(4C_A\epsilon\int_0^t (1+s)^{-\frac{n-1}{2}}\ ds\right) \\ &\le 8A(0) \\ &\le \frac{8A\epsilon}{16} \\ &= \frac{A\epsilon}{2} \end{aligned}$$

Thus we have that E is nonempty, relatively closed, and relatively open, so that E must be all of $[0, T_*)$, as desired. □

7.2.3 Brief Discussion of $n \le 3$

In lower dimensions we do not have global existence. However, we do have a result about long time existence, with $T_* \to \infty$ for $\epsilon \to 0$.

Theorem 7.6. *If $n \le 3$, $f, g \in C_c^\infty$ and $\epsilon > 0$ is sufficiently small, then there exists a solution u to (23) on $[0, T_\epsilon) \times \mathbb{R}^n$ with*

$$\begin{cases} e^{c/\epsilon} & n = 3 \\ \left(\frac{c}{\epsilon}\right)^2 & n = 2 \\ \frac{c}{\epsilon} & n = 1 \end{cases}$$

for some small constant c.

We do not prove this theorem, but note that revisiting the proof of local existence above we see that we needed

$$\epsilon\int_0^{T_\epsilon} \frac{1}{(1+t)^{\frac{n-1}{2}}}\ dt$$

to be small. T_ϵ is chosen to satisfy this requirement.
Since in the case of $n = 3$, T_* increases exponentially to ∞ as $\epsilon \to 0$, we say that for $n = 3$, there exist almost global solutions to the quasilinear wave equation.

7.2.4 Criterion for Global Existence for $n = 3$

In our proof of global existence, the integrability of $(1+t)^{-\frac{n-1}{2}}$ was crucial. In the case of $n = 3$, we lose integrability. In fact, one cannot say in general that there exist global solutions to the quasilinear wave equation for $n = 3$.

One example, due to John, is given by the equation

$$\Box u = (\partial_t u)^2$$

with $(t, x) \in R_+ \times \mathbb{R}^3$. In this case, every non-trivial C^3 solution with compactly supported Cauchy data blows up in finite time. The reader is referred to John for details.

On the other hand, consider the equation

$$\begin{cases}\Box u = (\partial_t u)^2 - |\nabla_x u|^2 \\ u(0,\cdot) = \epsilon f \quad \partial_t(0,\cdot)\end{cases}$$

with $f, g \in C_c^\infty$. In this case there exist global solutions. This result is due to Nirenberg. To see this result, set $v(t,x) = 1 - e^{-u(t,x)}$. Then v solves the linear Cauchy problem

$$\Box v = 0; \qquad v(0,\cdot) = 1 - \epsilon^{-\epsilon f}; \quad \partial_t v(0,\cdot) = \epsilon g e^{-\epsilon f}$$

and thus v exists globally. Since $u = -\log(1-v)$ we have that u is a global solution provided $\|v\|_{L^\infty} \leq 1$. Our earlier results on the linear wave equation guarantee that this is the case if the initial data is sufficeintly small. Since $v(0,\cdot) = O(\epsilon)$ we see that this is obtainable for sufficiently small ϵ.

We also note without commentary that for $\Box u = |\nabla_x u|^2$ there does not exist a global solution for $n = 3$.

The natural question, then, is what condition is needed for the linear structure to guarantee global existence for $n = 3$? The answer comes from what is called the null condition.

We first discuss some intuition behind the null condition. Recall that the integrability of the gradient is a key piece to existence of solutions. We have that the gradient decays like $\frac{1}{t^2}$ away from the light cone, which is sufficient decay. Even on the light cone, there is only one problematic direction. The angular derivative and the derivatives in the direction of the light cone $(\partial_r + \partial_t)$ both decay like $\frac{1}{t^2}$. In the direction $(\partial_r - \partial_r)$, which is perpendicular to the light cone, we only have $\frac{1}{t}$ decay. Ultimately we want to restrict to the case that each quadratic interaction contains at most one term in this "bad" direction. The connection between this constraint and the null condition is not trivial. However, one can follow the proof that the null condition allows for necessary decay to see that it is equivalent to this constraint. We will not discuss this further in these notes, and doing so is left as an exercise for the reader.

To motivate the null condition more explicitly, we restrict to the similinear equation for systems

$$\begin{cases}\Box u^I = F^I(u, u') \\ u(0,\cdot) = \epsilon f; \quad u'(0,\cdot) = \epsilon g\end{cases}$$

for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^3$ and $I = 1, \ldots, N$. Assume $F(0,0) = 0 = dF(0,0)$. We write

$$F(z) = F_0(z) +)(|z|^3) \qquad F_0^I(z) = \sum_{|\alpha|=2} \frac{1}{\alpha!}\left(\partial^\alpha F^I(0)\right) z^\alpha \qquad z \in \mathbb{R}^{N+(1+3)N}$$

$$F_0^I = F_0^I(u') = \sum_{L,M=1}^{N} \sum_{\alpha\beta=0}^{3} a_{LM}^{\alpha\beta I} \partial_\alpha u^L \partial_\beta u^M$$

We'll say that the null condition holds if

$$\sum_{\alpha\beta=0}^{3} a_{LM}^{\alpha\beta I}\xi_\alpha\xi_\beta = 0 \qquad \forall\xi \text{ s.t. } \xi_0^2 = \xi_1^2+\xi_2^2+\xi_3^2$$

we say that vectors ξ satisfying $\xi_0^2 = \xi_1^2+\xi_2^2+\xi_3^2$ are null.
The following lemma categorizes bilinear forms satisfying the null condition.

Lemma 7.7. *Let B be a bilinear form on$\mathbb{R}^4\times\mathbb{R}^4$ such that $B(\xi,\xi)=0$ for all null ξ. Then B is a linear combination of the null forms*

$$Q_0(\xi,\eta) = \xi_0\eta_0 - \sum_{j=1}^{3}\xi_j\eta_j \qquad Q_{\mu\nu}(\xi,\eta) = \xi_\mu\eta_\nu - \xi_\nu\eta_\mu$$

We give only a sketch of a proof of the lemma.

Proof. We write $B(\xi,\xi) = \xi^T A\xi$ and decompose A into symmetric and antisymmetric pieces. With $A_s = \frac{A+A^T}{2}$ and $A_a = \frac{A-A^T}{2}$, we have $A = A_s + A_a$. For A_s we note that $\xi^T A_s\xi = 0$ for all null ξ and denote $A = (a^{\mu\nu})$. Use the following values for ξ:

$$\begin{array}{ll} (\pm1,1,0,0) & (\sqrt{2},1,1,0) \\ (\pm1,0,1,0) & (\sqrt{2},1,0,1) \\ (\pm1,0,0,1) & (\sqrt{2},0,1,1) \end{array}$$

to show $A_s = a^{00}\mathrm{diag}(1,-1,-1,-1)$.
Working with A_a is easier and is left as an exercise. We then have

$$B = a^{00}Q_0 + \frac{1}{2}\sum_{0\le\mu<\nu\le3}\left(a^{\mu\nu}-a^{\nu\mu}\right)Q_{\mu\nu}$$

□

Next we bound null forms as in the above lemma.

Lemma 7.8. *Let Q be one of the null forms*

$$Q_0(v,w) = \partial_0 v\partial_0 w - \sum_{i=1}^{3}\partial_i v\partial_i w \qquad Q_{\mu\nu} = \partial_\mu v\partial_\nu w - \partial_\mu w\partial_\nu v$$

Thin if $t>0$,

$$|Q(v,w)(t,x)| \le \frac{C}{1+t+|x|}\sum_{|\alpha|=1}|\Gamma^\alpha v|\sum_{|\alpha|=1}|\Gamma^a w|$$

where, as before, $\Gamma = \{\partial,\Omega,S\}$.

Proof. WLOG assume $t + |x| < 1$. For $1 \leq i < j \leq 3$ we have

$$\partial_i = \sum_1^3 \frac{x_i x_j}{r^2}\partial_j + \sum_1^3 \frac{x_j \Omega_{ij}}{r^2}$$

We calculate

$$\begin{aligned} Q_{ij}(v,w) &= \frac{1}{t}\left[\partial_t v \Omega_{ij} w + \Omega_{oi} v \partial_j w - \Omega_{0j} v \partial_i w\right] \\ &= \frac{1}{|x|}\left[\partial_r v \Omega_{ij} w + \sum_{k=1}^3 \frac{x_k}{|x|}\Omega_{ik} v \partial_j w - \sum_{k=1}^3 \frac{x_k}{|x|}\Omega_{jk} v \partial_i w\right] \end{aligned}$$

$$\begin{aligned} Q_{0j}(v,w) &= \frac{1}{t}\left(\partial_t v \Omega_{0j} w - \partial_t w \Omega_{0j} v\right) \\ &= \frac{1}{|x|}\left(\frac{x_j}{|x|}(\Omega_r v \partial_r w - \partial_r v \Omega_r w) + \partial_t v \sum_{k=1}^3 \frac{x_k}{|x|}\Omega_{jk} w - \partial_t w \sum_{k=1}^3 \frac{x_k}{|x|}\Omega_{jk} v\right) \end{aligned}$$

where

$$\Omega_r = \sum_{j=1}^3 \frac{x_j}{|x|}\Omega_{0j} = r\partial_t + t\partial_r$$

$$\begin{aligned} Q_0(v,w) &= \frac{!}{t}(\partial_t v S w - \sum_{i=1}^3 \Omega_{0i} v \partial_i w) \\ &= \frac{1}{|x|}\left(\Omega_r v \partial_t w - \partial_r v S w - \sum_{i=1}^3 \Omega_{ik} w\right) \end{aligned}$$

□

This concludes our discussion on the techniques used in using the null condition to obtain global existence for $n = 3$.

7.2.5 Almost Global Existence for $n = 3$

We will sketch a proof of the following theorem

Theorem 7.9. *Let $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$ and consider the equation*

$$\begin{cases} \Box u = F(u') \\ u(0,\cdot) = \epsilon f; \quad \partial_t u(0,\cdot) = \epsilon g \\ f, g \in C_c^\infty \\ \quad F(0) = dF(0) = 0 \end{cases} \tag{28}$$

There exists a solution u on $[0, T_\epsilon] \times \mathbb{R}^3$ with $T_\epsilon = \exp\left(\frac{c}{\epsilon}\right)$.

Recall that for $n \geq 3$ we have the local energy estimate

$$\sup_R R^{-1/2}\|u'\|_{L^2_{t,x}([0,T]\times\{|x|\leq R\})} + \|u'\|_{L^\infty_t L^2_x([0,T]\times\mathbb{R}^n)} \lesssim \|u'(0,\cdot)\|_{L^2} + \int_0^T \|\Box u(s,\cdot)\|_{L^2}\, ds$$

We claim that the following inequality holds:

$$(\log(2+T))^{-1/2}\|\langle x\rangle^{-1/2}u'\|_{L^2_{t,x}([0,T]\times\mathbb{R}^n)} + \|\langle x\rangle^{-1/2-\delta}u'\|_{L^2_{t,x}([0,T]\times\mathbb{R}^n)} \lesssim \|u'(0,\cdot)\|_{L^2} + \int_0^T \|\Box u(s,\cdot)\|_{L^2}\, ds$$

Proof. We prove the bound for each term individually.
First term

- $|x| \geq T$

$$\|\langle x\rangle^{-1/2}u'\|_{L^2_{t,x}([0,T]\times\{|x|\geq T\})} \lesssim \|u'\|_{L^\infty_t L^2_x} \lesssim RHS$$

- $|x| < T$

$$\begin{aligned}\|\langle x\rangle^{-1/2}u'\|_{L^2_{t,x}([0,T]\times\{|x|\leq T\})} &\lesssim \sum_{j=0}^{\log(2+T)} \|\langle x\rangle^{-1/2}u'\|_{L^2_{t,x}([0,T]\times\{\langle x\rangle\approx 2^j\})} \\ &\lesssim \sum_{j=0}^{\log(2+T)} (RHS)^2 \\ &= (\log(2+T))(RHS)^2\end{aligned}$$

Second term

$$\|\langle x\rangle^{-1/2-\delta}u'\|_{L^2_t L^2_x} = \sum_{j=0}^{\infty} 2^{-j\delta 2}\left[2^{-j}\|u'\|_{L^2_{t,x}([0,T]\times\{\langle x\rangle\approx 2^j\})} \lesssim RHS\right]$$

□

We establish one more estimate before sketching the proof of Theorem 7.9.

Lemma 7.10. *Let $h \in C^\infty(\mathbb{R}^n)$ and $R \geq 1$. Then*

$$\|h\|_{L^\infty(\frac{R}{2}<|x|<R)} \lesssim R^{-\frac{n-1}{2}} \left\| Z^{\leq \lfloor \frac{n+2}{2}\rfloor} h\right\|_{L^2(\frac{R}{4}<|x|<2R)}$$

with $Z \in \{\partial, \Omega_{ij}\}$.

Proof. We have

$$\|h\|_{L^\infty} \lesssim \|\partial^{\frac{n+2}{2}} h\|_{L^2}$$

by Sobolev inequalities. Thus

$$\|h\|_{L^\infty(|x|\approx R)} \lesssim \frac{1}{R^{\frac{n-1}{2}}} \left\| Z^{\leq \lfloor \frac{n+2}{2}\rfloor} h\right\|_{L^2}$$

□

Finally we sketch the proof of Theorem 7.9. We follow the standard structure of defining u_j iteratively, establishing boundedness, then showing that the sequence is Cauchy.

Proof. of Theorem 7.9 Define $u_{-1} \equiv 0$. Let u_j solve

$$\begin{cases} \Box u_j = F(u'_{j-1}) \\ u_j(0,\cdot) = \epsilon f; \quad \partial_t u_j(0,\cdot) = \epsilon g \end{cases}$$

First we establish boundedness. Specifically, we wish to show there exists an A so that for $T < T_\epsilon = \exp\left(\frac{c}{\epsilon}\right)$

$$A_j(T) := (\log(2+T))^{-1/2} \|\langle x\rangle^{-1/2} Z^{\leq 10} u'_j\|_{L^2_{t,x}([0,T]\times\mathbb{R}^n)} + \|\langle x\rangle^{-1/2-\delta} Z^{\leq 10} u'_j\|_{L^2_{t,x}([0,T]\times\mathbb{R}^n)} \leq A\epsilon$$

The commutator relations $[\Box, \partial] = 0 = [\Box, \Omega_{ij}]$ along with local energy give

$$\begin{aligned}(\log(2+T))^{-1/2} \|\langle x\rangle^{-1/2} Z^{\leq 10} u'_j\|_{L^2_{t,x}([0,T]\times\mathbb{R}^n)} + &\|\langle x\rangle^{-1/2-\delta} Z^{\leq 10} u'_j\|_{L^2_{t,x}([0,T]\times\mathbb{R}^n)} \\ &\leq \frac{A}{2}\epsilon + C\int_0^T \|Z^{\leq 10} F(u'_{j-1})\|_{L^2}\, ds\end{aligned}$$

Thus $A_0(T) \leq \frac{A}{2}\epsilon$ for all T.
For the inductive hypothesis, assume $A_{j-1} \leq A\epsilon$ for all $T \leq T_\epsilon$. It suffices to show

$$C\int_0^T \|Z^{\leq 10} F(u'_{j-1})\|_{L^2}\, ds \leq \frac{A}{2}\epsilon$$

$$\begin{aligned}\|Z^{\leq 10} F(u'_{j-1})\|^2_{L^2} &\lesssim \left\| |Z^{\leq 5} u'_{j-1}| |Z^{\leq 10} u'_{j-1}| \right\|^2_{L^2} \\ &\lesssim \sum_j \|Z^{\leq 5} u'_{j-1}\|_{L^\infty(2^{j-1}\leq\langle x\rangle\leq 2^j)} \|Z^{\leq 10} u'_{j-1}\|_{L^2(2^{j-1}\leq\langle x\rangle\leq 2^j)} \\ &\lesssim \sum_j \left(2^{-j/2}\|Z^{\leq 7} u'_{j-1}\|_{L^2(2^{j-2}\leq\langle x\rangle\leq 2^{j+1})}\right)\left(2^{-j/2}\|Z^{\leq 10} u'_{j-1}\|_{L^2(2^{j-1}\leq\langle x\rangle\leq 2^j)}\right) \\ &\lesssim \sum_j \|\langle x\rangle^{-1/2} Z^{\leq 10} u'_{j-1}\|^2_{L^2} \\ &\leq C(\log(2+T))(A_{j-1}(T))^2\end{aligned}$$

Thus we are done if $c(\log(2+T))(A\epsilon)^2 \leq \frac{A\epsilon}{2}$, which is true if $2C\log(2+T)A\epsilon \leq 1$, which is true if $T \leq T_\epsilon$.
Note that we have swept some of the details under the rug. But the components needed are there. We ommit the proof that the sequence is Cauchy, but note that it follows the same structure as in previous proofs. □