# *Research Statement*

for Katrina Morgan

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I am interested in problems in partial differential equations, spectral theory, nonlinear waves, and global analysis. I have used spectral and microlocal techniques to establish the relationship between wave decay and underlying metric behavior at spatial infinity for a broad class of geometries in [Mor] and Klainerman vector field methods to prove global existence of solutions to systems of quasilinear wave equations in [MetMor].

Currently, I am investigating the effect of geodesic trapping (i.e. null geodesics which remain within a compact region) on a dispersive estimate called *local energy decay*. This estimate informs much of my work, so I will provide heuristics before discussing my research in detail.

**Local Energy Decay**

The *local energy decay* estimate has proven to be a powerful tool for studying wave behavior on asymptotically flat spacetimes. The estimate holds if the underlying geometry allows waves to spread out enough so that energy within compact spatial regions decays quickly enough to be integrable in time. It has been used to establish e.g. Strichartz estimates (global, mixed norm estimates) in [MetTat, Toh, MaMeTaTo] and pointwise estimates in [Tat, Mor, DafRod, MeTaTo] (among others).

Local energy decay is sensitive to (i) geodesic trapping and (ii) resolvent behavior in the lower half plane and on $\mathbb{R}$. Trapping occurs when a null geodesic remains within a compact region. If the trapping is unstable, a weaker form of the estimate (i.e. *weak local energy decay*) may hold. A discussion of resolvent behavior is reserved for section 2.

## 1. Wave decay on asymptotically flat spacetimes

The wave operator on a Lorentzian metric $g$, denoted $\Box_g$, is given by the d'Alembertian, which generalizes the Laplace operator. The flat Minkowski metric in $(1+3)$ dimensions is given by $m = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2$. The d'Alembertian on Minkowski spacetime reduces to the flat wave operator $-\partial_t^2 + \Delta$, where $\Delta = \sum_{i=1}^{3} \partial_i^2$ is the flat Laplacian. A metric $g = g_{\alpha\beta} d\alpha d\beta$ is said to be *asymptotically flat* if $\lim_{|x|\to\infty} g_{\alpha\beta} = m_{\alpha\beta}$.

In [Mor] I quantified the relationship between pointwise wave decay and the rate at which a geometry becomes flat. My result interpolates between two known results for $(1+3)$ dimensional spacetimes with time independent metrics:

| | Metric Behavior | Wave decay |
|---|---|---|
| [Tat]: | $g = flat + \mathcal{O}(r^{-1})$ | $\lvert u(t,x)\rvert \lesssim t^{-3}$ |
| **[Mor]**: | $g = flat + \mathcal{O}(r^{-k})$ | $\lvert u(t,x)\rvert \lesssim t^{-\lfloor k\rfloor -2}$ |
| Sharp Huygens': | $g = flat$ | $\lvert u(t,x)\rvert \lesssim t^{-\infty}$ |

The decay rates found in [Tat] and [Mor] hold for solutions to $\Box_g u = 0$ with smooth, compactly supported Cauchy data, ***if the weak local energy decay estimate holds on $g$.***

The problem studied in [Tat] was motivated by Price's Law, a conjecture from physics predicting a $t^{-3}$ decay rate for waves on the spacetime geometry created by a non-rotating black hole. The weak local energy decay estimate for this geometry was previously established in [MaMeTaTo]. Weak local energy decay handles geodesic trapping and provides basic low frequqncy information. The metric behavior at spatial infinity is then left to consider.

My research shows that the metric behavior at spatial infinity ultimately determines the rate of pointwise wave decay. I consider the homogeneous Cauchy problem

$$(\Box_g + V)u = 0, \quad u(0,\cdot) = u_0, \quad \partial_t u(0,\cdot) = u_1, \qquad u_0, u_1 \in C_0^\infty \tag{1}$$

where $V$ is a scalar potential of the form $V(x) = V^{sr}(x) + V^{lr}(|x|) \in \mathcal{O}(r^{-k-2-\epsilon}) + \mathcal{O}(r^{-k-2})$.

**Theorem 1.1.** [Mor] *Let $k \geq 1$. Let $g = m + g^{sr} + g^{lr} = m + \mathcal{O}(r^{-k-\epsilon}) + \mathcal{O}_{rad}(r^{-k})$ where the* rad *subscript indicates $g^{lr}$ is spherically symmetric. Assume $u$ solves (1). If the evolution (1) satisfies uniform energy bounds and the weak local energy decay estimate, then the following estimates hold:*

$$|u(t,x)| \leq C_x t^{-\lfloor k \rfloor - 2}, \qquad |\partial_t u(t,x)| \leq C_x t^{-\lfloor k \rfloor - 3} \tag{2}$$

The result is stated in normalized coordinates, chosen so that

$$\Box_g + V = -\partial_t^2 + \Delta + \partial_t P^1 + P^2 =: P$$

where $P^1$ and $P^2$ are respectively first and second order spatial operators with coefficients which decay at a rate which depends on $k$.

The case where $k = 1$ was studied in [Tat] and there the estimates are believed to be sharp. I intend to prove that the pointwise decay rates above are optimal for all integer values of $k$.

**Problem 1.2.** *The estimates (2) are sharp for $k \in \mathbb{N}$.*

A similar result was shown in [SzBiChRo] for the flat wave operator with a polynomially decaying scalar potential, including an argument establishing sharpness of the pointwise wave decay. In [Tat] and [Mor], the final decay rates are ultimately limited by a similar scalar term, which arises due to the $g^{lr}$ piece when writing the operator $(\Box_g + V)$ in normalized coordinates. I plan to adapt the sharpness argument in [SzBiChRo] to prove the conjecture.

I expect the pointwise decay rate should be continuous in $k$, so the above result is almost certainly not sharp for $k \notin \mathbb{N}$.

**Problem 1.3.** *What are the sharp estimates for $k \notin \mathbb{N}$?*

The techniques I used to prove Theorem 1.1 are not sensitive to non-integer values of $k$. In [BonHaf2, BonHaf3] the authors obtain a local decay rate (analogous to pointwise decay) for waves which depends continuously on the rate at which the background geometry becomes flat. They consider product manifolds rather than the full Lorentzian perturbations considered here. I will investigate if their techniques are useful for obtaining more meaningful information for non-integer values of $k$.

I plan to study the following extensions of my work in [Mor]:

**Problem 1.4.** *Establish pointwise decay rates for spacetimes as in Theorem 1.1 for $d \geq 4$ spatial dimensions.*

**Problem 1.5.** *Establish pointwise decay rates for nonstationary spacetimes analogous to those in Theorem 1.1.*

**Problem 1.6.** *Establish pointwise decay rates for spacetimes as in Problem 1.5 for $d \geq 4$ spatial dimensions.*

The 3-dimensional nonstationary case for $k = 1$ was studied in [MeTaTo]. The techniques used in [Mor] are not applicable to time dependent spacetimes as they rely on taking the Fourier transform in time. I propose to adapt the techniques in [MeTaTo] first to $k > 1$ then to higher dimensions.

## 2. Resolvent bounds and local energy decay

The utility of weak local energy decay in proving Theorem 1.1 is its connection to the resolvent.

**Definition 2.1.** *The* ***resolvent****, $R^g_\tau$, associated to $\Box_g = P$ in a fixed coordinate system is given by $R^g_\tau := \hat{P}^{-1}$ when the inverse exists. $\hat{P}$ is the image of $P$ under the time Fourier Transform.*

Note that the resolvent associated to $\Box_m = -\partial_t^2 + \Delta$ is the standard resolvent of the free Laplacian: $R^m_\tau = (\tau^2 + \Delta)^{-1}$. In the coordinate system used in Theorem 1.1, the resolvent associated to $\Box_g$ is a $\tau$-dependent perturbation of the resolvent of the free Laplacian:

$$R^g_\tau = (\tau^2 + \Delta + \tau P^1 + P^2)^{-1} \tag{1}$$

. If $u$ solves (1), then the above definition allows us to connect $\hat{u}(\tau)$ to the resolvent applied to the initial data.

It was shown in [MeStTa] that the local energy decay estimate implies that the resolvent exists for $\Im\tau < 0$ and that it can be extended to $\tau \in \mathbb{R}$. This holds for non-trapping, stationary, asymptotically flat backgrounds, regardless of the rate toward flat. The picture is more complicated for the weak local energy decay estimate, where trapped geodesics may occur.

In [Mor] the resolvent is initially shown to exist for $\Im\tau < 0$. It is then extended to the real line by commuting $\hat{P}$ with vector fields and using the weak local energy decay estimate to obtain resolvent bounds. The commutator structure depends on the coefficients of $\hat{P}$, which in turn depend on the rate at which the geometry tends toward flat. The zero resolvent is particularly sensitive to the far-away metric behavior.

I will study the following problem, which considers the minimal rate toward flat needed to be able to extend the resolvent to the real line using weak local energy decay.

**Problem 2.2.** *Let $g$ be a spherically symmetric perturbation of Minkowski space-time which tends toward flat at a rate of $r^{-\epsilon}$ for $0 < \epsilon < 1$. If the weak local energy decay property holds, then the resolvent extends to $\tau \in \mathbb{R} \setminus \{0\}$.*

The precise statement is based on a preliminary review of the calculations in [Mor]. It appears that perturbations decaying more slowly than $\mathcal{O}(r^{-1-\epsilon}) + \mathcal{O}_{rad}(r^{-1})$ would preclude extending $R_\tau$ to 0 using weak local energy. Only the spherically symmetric perturbation seems to be able to decay more slowly than $r^{-1}$ while preserving the continuation of $R_\tau$ to nonzero real $\tau$. Working with spherical symmetry has the benefit that $\hat{P}$ will commute with rotational vector fields.

Problem 2.2 and the preceding discussion leads to an extension of [Mor].

**Problem 2.3.** *Can we study long-time wave behavior on asymptotically flat spacetimes of the form $g = m + \mathcal{O}(r^{-\epsilon})$ for $0 < \epsilon < 1$ either by replacing the weak local energy decay estimate with the local energy decay estimate or by restricting to spherically symmetric perturbations?*

The key here is whether or not we can extend the resolvent to the real line under the given assumptions. Weak local energy decay likely presents a problem at 0, even if we assume spherical symmetry, but the local energy decay estimate would allow this step.

## 3. Nonlinear wave equations

Nonlinear wave equations are of the form $(-\partial_t^2 + \Delta)u = F(\nabla^{\le M} u)$ where $F$ is a nonlinear function which may depend on $u$ and its derivatives. Global existence of solutions in $n$ spatial dimensions for small data was established in [Hor] for nonlinearities $F(u, u', u'')$ which vanish to second order and satisfy $\partial_u^2 F(0,0,0) = 0$. The inclusion of $u$ in the nonlinearity creates challenges, especially in smaller spatial dimensions.

Global existence of soultions to quasilinear wave equations exterior to a star-shaped obstacle (denoted $\mathcal{K}$) in $n = 4$ spatial dimensions for nonlinearities as in [Hor] and small initial data was established in [MetSog]. The inclusion of an obstacle complicates the picture due to the presence of a boundary. In the presence of an obstacle, the initial data must satisfy certain *compatibility conditions* on $\partial\mathcal{K}$.

The results of both [Hor] and [MetSog] relied on rewriting the lowest order terms $u\partial_\alpha u$ as $\frac{1}{2}\partial_\alpha(u^2)$, which does not work for systems. Because the initial data is small, the lower order terms are hardest to handle. In [MetMor] we alter the arguments of [MetSog] in order to establish global existence of solutions to quasilinear systems of the form

$$\begin{cases} \Box_{C_I} u^I = F^I(u, u', u''), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^4 \setminus \mathcal{K}, \ I = 1, 2, \dots, M \\ u^I(t,\cdot)|_{\partial\mathcal{K}} = 0 \\ u^I(0,\cdot) = f^I, \quad \partial_t u^I(0,\cdot) = g^I \end{cases} \tag{1}$$

where $\Box_{C_I} = \partial_t^2 - c_I^2\Delta$ is the wave operator with speed $c_I$ and $\mathcal{K}$ is a compact, star-shaped obstacle with smooth boundary.

Our argument uses vector field methods and relies on a lemma allowing a derivative to be exchanged for better decay within the light cone.

**Theorem 3.1.** *Assume $F^I(u, u', u'')$ vanishes to second order and satisfies $\partial_u^2 F^I(0,0,0) = 0$. Assume $f^I, g^I$ satisfy the necessary compatibility conditions. If $f^I$ and $g^I$ are sufficiently small then (1) has a unique global solution.*

A natural problem to consider next is the case of $n = 3$ spatial dimensions, where only almost global existence is expected to hold. Solutions are said to exist almost globally if there is a unique solution for $t \in [0, T]$ where $T$ grows exponentially as the size of the initial data decreases.

**Problem 3.2.** *Obtain almost global existence for systems as in* (1) *for 3 spatial dimensions.*

In the case of 3 spatial dimensions, the numerology of the exponents in one of the estimates used to prove Thm 3.1 just misses integrability. I plan to modify the argument in [MetMor] to solve the above problem.

## 4. Geometry of (weak) local energy decay

### 4.1. **The effect of geodesic trapping on weak local energy decay.**

The presence of trapping necessitates a derivative loss in the weak local energy decay estimate. The precise nature of the loss depends on the nature of the trapping. A similar picture holds for the local smoothing estimate for the Schrödinger equation, which (without loss) says that solutions gain a $\frac{1}{2}$ power of regularity compared to the data. Some known results are summarized in the table below:

| Trapping | Loss - Weak LED | Loss - Local Smoothing |
|---|---|---|
| nondegenerate, unstable | logarithmic[1] | logarithmic[3] |
| degenerate, unstable | polynomial[2] | polynomial[4] |

([1]) was established in [MeMaTaTo]. ([2]) was established in [BoChMePe]. There are many examples of ([3]) including [Bur, Chr, Dat, WunZwo]. ([4]) was established in [ChrWun], which provided the first example of polynomial loss for the local smoothing estimate.

My current work seeks to establish a polynomial loss in weak local energy decay for a background with mixed unstable and inflection-transmission trapping.

**Problem 4.1.** ***(In Progress)*** *Prove weak local energy decay estimates with polynomial loss for the wave equation on a surface of revolution where the generating curve has a minimum at $x = 0$ and an inflection point at $x = 1$.*

One formulation of weak local energy decay uses a smooth cutoff to remove a neighborhood of the trapped geodesic. The lossy estimate can then be obtained from the cutoff formulation. For local smoothing, there is a similar picture, and the resulting behavior depends on the size of the cutoff necessitated by the trapping. See e.g. [DatJin]. Whether a corresponding statement holds for weak local energy decay is currently unknown. I plan to work on the following problem:

**Problem 4.2.** *If trapping necessitates a large enough cutoff, does weak local energy decay break? If not, how will the amount of loss be affected?*

### 4.2. **Geometric classification of local energy decay.**

In [MeStTa] the authors showed that for stationary nontrapping backgrounds, the local energy decay estimate holds if and only if the resolvent exhibits nice behavior on the real line and in the lower half plane. The local energy decay estimate is geometric in that it depends on how much dispersion is allowed by the background geometry. However, it is not clear how the spectral conditions required for local energy decay relate to geometric properties.

I propose to study the following in order to fully classify the geometric conditions necessary for local energy decay to hold.

**Problem 4.3.** *Establish geometric conditions sufficient to obtain local energy decay estimates on non-trapping, time independent, asymptotically flat spacetimes.*

Trapping is a high frequency phenomenon, so much of the high frequency resolvent behavior is handled by the non-trapping assumption. It is the low frequency analysis (particularly existence of the resolvent at 0 energy) that requires careful analysis.

Note that for non-product manifolds the resolvent is as in (1). In the case of product manifolds, the resolvent is easier to work with because it no longer includes the $\tau P^1$ term. Thus, I plan to begin by considering product manifolds.

Spectral theory for perturbations of the Laplacian guarantees that the resolvent exists in the lower half plane for asymptotically flat product manifolds. Results in [BonHaf1] and [Bou] establish low frequency resolvent estimates on such manifolds for $\tau \notin \mathbb{R}$. These estimates hold uniformly as $\tau \to \mathbb{R}$. The recent work [Vas] establishes the resolvent can be extended to zero, but the target and image function spaces must be extended and restricted, respectively. The function spaces on which the zero resolvent exists will affect whether the local decay estimate can be recovered.

We note that resolvents including the $\tau P^1$ term in Definition (2.1) are also studied in [Vas]. This provides a starting point to consider the non-product manifold case. While the geometric classification of weak local energy decay would be interesting to study, a result analogous to that in [MeStTa] would first need to be established for the above approach to work.

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