Special Relativity

The following problems should all be done as a group. Make sure to communicate with each other the reasoning for your solutions. If different group members have different answers, try to talk it through as a group and reach a consensus. Periodically check with your team leader that your solutions are correct. If they are not, go back and try to fix them. The problems are designed to build off one another, so be sure to go through them in order.

1 Introduction and Warm-Up

1.1 *From Terence Tao's Math Circle Special Relativity Notes*
Alice is trying to get from one end of a large airport to another. The airport is 500 meters long and she walks at a pace of 1 meter/second. In the middle of the airport, there is a moving walkway that is 100 meters long, and moves in the direction Alice travels at a rate of one meter per second. This means that if Alice stands still while on the moving walkway, she travels at a speed of one meter per second and if she continues to walk while on the walkway, she travels at a speed of two meters per second. However, Alice's shoelaces are untied and she must tie them before she reaches the other end of the airport. Alice can tie her shoes (pretty quickly, at that) and it will take her ten seconds to do so. She wants to get to the other end of the airport as soon as possible, so she must decide whether she will tie her shoes before she reaches the moving walkway, while on the moving walkway, or after she has passed the moving walkway.

(a) Draw spacetime diagrams to illustrate each of the three different choices Alice can make.

(b) What should Alice do, i.e., when would be the optimal time for Alice to tie her shoes?

(c) Does the answer to (b) change if one changes the numbers in the exercise (e.g. length of the walkway, Alice's walking speed, speed of the walkway, time it takes Alice to tie her shoes)?

You have probably heard the term "light-year" at some point. This refers to how far light travels in the course of a year. Note that a light-year is a unit of *distance*. Similarly, a **light-second** is a unit of *distance* that indicates how far light travels in one second. A second is a unit of *time*, but a light-second is a unit of *distance*.

We can also use this method to define units of time. A **light-meter** is the amount of time it takes for light to travel one meter. Note that while a meter is a unit of *distance*, a light-meter is a unit of *time*.

1.2 Perform the following unit conversions, using $c \approx 3 \times 10^8$ meters/second.

(a) 1 light-second = ? meters

(b) 1 meter = ? light-seconds

(c) 1 light-meter = ? seconds

(d) 1 second = ? light-meters

(e) 30 miles/hour = ? meters/light-meter

(f) $c \approx 3 \times 10^8$ meters/second = ? meters/light-meter

2 The Problem with Classical (Newtonian) Mechanics

In classical mechanics, speeds in the same direction are additive, and time and distance are absolute quantities for all observers. Objects behave in an intuitive way. However, we will see that as speeds approach the speed of light, these mechanics fail to explain the observation that the speed of light is the same for all observers.

2.1 Suppose Alice is stationary and a light-source is moving towards her. At the moment the light source is 1 light-second in front of her, it emits a flash of light (remember a light-second is a unit of distance). How fast does Alice see the flash of light moving if the light source is travelling at the following speeds?

(a) $0.9 \times c$ (where $c \approx 3 \times 10^8$ meters/second is the speed of light)

(b) $0.75 \times c$

(c) $0.5 \times c$

(d) 340m/s (the speed of sound)

(e) 12.4m/s (the world record for fastest human footspeed)

(f) According to the second postulate of Special Relativity, Alice will see the flash of light move at the speed of light no matter how quickly she is moving. What can you say about the classical mechanical perspective's ability to explain this phenomenon?

2.2 Alice is floating in space. A spaceship passes in front of Alice at $0.75c$ (0.75 times the speed of light). Inside the spaceship is an alien named Barbara and two mirrors on the floor and ceiling separated one light-second apart. There is a photon (light particle) bouncing back and forth between the mirrors. At the exact moment that the mirrors pass Alice, the photon is at the bottom mirror. Take this instant to be time $t = 0$.

(a) We will first consider Barbara's point of view.

   (i) From Barbara's point of view does the photon have vertical motion? Does the photon have horizontal motion?

   (ii) What is the position of the photon after 1 second? What about after $2, 3,$ and 4 seconds?

   (iii) Draw the path of the photon from Barabara's point of reference (take the origin to be the bottom of the mirror). Note that this is NOT a spacetime diagram. Your axes should be the horizontal and vertical directions. Use light-seconds as your unit of distance.

(b) We will now consider Alice's point of view.

   (i) From Alices point of view does the photon have vertical motion? Does the photon have horizontal motion?

   (ii) What is the position of the photon after 1 second? What about after $2, 3,$ and 4 seconds?

   (iii) Draw the path of the photon from Alice's point of reference (take the origin to be the bottom of the mirror). Note that this is NOT a spacetime diagram. Your axes should be the horizontal and vertical directions. Use light-seconds as your unit of distance.

(c) After 4 light-meters how far does the photon travel from Barbara's point of view? From Alice's?

(d) What is the photon's speed from Barbara's point of view? From Alice's?

(e) What equation did you use to find the speeds in part (d)? Thinking about the variables you use in this equation, what two measurements would need to change to make the assumption that the speed of light is the same observers hold? In what direction would they need to change (i.e. increase or decrease)?

(f) Find a general equation for the photon's speed from Alice's point of view if the spaceship moves at an arbitrary speed v light-seconds/second. (hint: it is moving at a constant speed, so you can just consider the distance it travels after 1 second)

(g) Calculate how fast Alice sees the photon move if the spaceship moves at the speeds given in $2.1d, e$.

3 A Closer Look at Classical Mechanics

3.1 Another Look at Classical Mechanics Again Alice is floating in space. This time her friend Barbara passes by her at $0.5c$. At the moment Barbara passes, they both emit a photon in the direction Barbara is moving. We first use the Classical Mechanical view. For this problem use light-meters as your unit of time and meters as your unit of distance.

(a) Draw a spacetime diagram from Alice's perspective showing the path of Alice, Alice's photon, Barbara, and Barbara's photon. (Remember we are now assuming Classical Mechanics)

(b) How fast does Alice think Barbara's photon is moving?

(c) Draw a spacetime diagram from Barbara's point of view showing the path of Alice, Alice's photon, Barbara, and Barbara's photon.

(d) How fast does Barbara think Alice's photon is moving?

(e) What aspect of the worldline of the photons indicates the speed?

3.2 Galilean Transformation We continue to consider the scenario from question 3.1. The goal in this problem is to understand the relationship between the diagrams for Alice and Barbara's reference frames. Let's say we label the space axis in Alice's frame x and we label the space axis in Barbara's frame x'. Similarly, we label the time axis in Alice's frame t and the time axis in Barbara's frame t'.

Recall that in Classical Mechanics we assume time is universal. This is an intuitive assumption. So intuitive that it may not feel like an assumption at all. When we consider Alice and Barbara's positions after 1 light-meter of time, we do not have to specify *whose* time.

(a) Fill in these coordinate systems with the spacetime diagrams you made in 3.1. Find the point $x = 0$meters, $t = 1$ light-meter on Alice's spacetime diagram. This corresponds to where Alice is after 1light-meter of time. (Why?)

(b) Find the point on Barbara's spacetime that corresponds to where Alice is after 1 light-meter of time. What are the t' and x' coordinates at this point?

(c) We are trying to find a formula to relate (x, t) to (x', t'). Because we are assuming time is the same for both Alice and Barbara, we have that $t' = t$. It is left to figure out how we can find x' given a point (x, t) in Alice's frame of reference. To do this, imagine that as Barbara is moving she is holding a long stick that is marked off in meters. The scenario at $t = t' = 0$ is shown below. Remember that Barbara thinks she's stationary, so this stick is a very reasonable measure of distance for her.

(i) Draw a picture like the one above showing where Alice, Barbara, and the stick are at time $t = t' = \mathbf{1}$. Consider a point 1 meter away from Alice ($x = 1$). How far in front of her does Barbara think this point is, using her measuring stick?

(ii) Draw a picture like the one above showing where Alice, Barbara, and the stick are at time $t = t' = \mathbf{2}$. Consider a point 1 meter away from Alice ($x = 1$). How far in front of her does Barbara think this point is, using her measuring stick?

(iii) Complete the following chart

(iv) Find an equation that allows you to find x' given x and t. In other words, find x' in terms of x and t.

(v) Can you generalize the formula from (iv) to give x' in terms of x and t if Barbara is moving at an arbitrary speed v?

4 A First Look at Spacetime Diagrams in Special Relativity

As we saw in the lecture, the second postulate of Special Relativity states that the speed of light is the same for all observers moving at a constant speed. In the last section we found that our intuitive assumptions about measuring distance and time were unable to make this postulate hold. In this section you will draw spacetime diagrams in which you start with the assumption that the speed of light is the same for all observers and explore some surprising results that follow from the second postulate of Special Relativity.

4.1 Alice is still floating in space. She again sees Barbara pass by her in a spaceship at $0.75c$. At the precise moment that the ship passes Alice, it sends out a flash of light in the same direction that the ship is moving. We will define the moment that this event occurs as time $t = 0$ light-meters and the location at which it occurs to be $x = 0$ meters.

(a) Note that we are using light-meters as our unit of time and meters as our unit of distance. What is the speed of light in these units? In other words $c =$? meters / light-meter.

(b) According to the second postulate of Special Relativity, your answer from part (a) is the speed at which light moves through all reference frames. What is the slope of the worldline describing the light's motion in any reference frame? How does the angle between the light's worldline and the x axis compare to the angle between the light's worldline and the t axis?

(c) Draw a spacetime diagram from Alice's perspective, assuming that she sees the flash moving at the speed of light. Include the worldlines for Alice, Barbara, and the light flash.

(d) Draw a spacetime diagram from Barbara's perspective, assuming that she sees the flash moving at the speed of light. Include the worldlines for Alice, Barbara, and the light flash.

4.2 Alice is still floating through space. This time Barbara and Cherly are on either side of her, each moving away from her in opposite directions at $0.4c$. At the moment that they are each one meter away, they both emit a flash of light back toward Alice.

(a) Draw a spacetime diagram from Alice's perspective, including worldines for Alice, Barbara, Cheryl, Barbara's light flash, and Cheryl's light flash. The point at which the path of Barbara's light flash crosses Alice's path represents when and where in spacetime Alice sees Barbara's flash reach to her. The same is true for Cheryl's light flash. Does Alice see Barbara's flash reach her before, after, or at the same time as Cheryl's light flash?

(b) Draw a spacetime diagram from Barbara's perspective, including worldlines for Alice, Barbara, Cheryl, Barbara's light flash, and Cheryl's light flash. Does Barbara see her light flash reach Alice before, after, or at the same time as she sees Cheryl's light flash reach Alice?

(c) Draw a spacetime diagram from Cheryl's perspective, including worldlines for Alice, Barbara, Cheryl, Barbara's light flash, and Cheryl's light flash. Does Cheryl see her light flash reach Alice before, after, or at the same time as she sees Barbara's light flash reach Alice?

4.3

Source: Taylor and Wheeler's "Spacetime Physics" Denise is floating in space. Alice, Barbara, and Cheryl are riding on a long spaceship that travels past Denise at a $0.9c$. Cheryl is in front, Alice is in the middle, and Barbara is at the rear. At the very instant Alice passes Denise it happens that two flash-bulb signals coming from Cheryl and Barbara reach Alice and Denise. Who emitted the signal first? Using only the fact that the speed of light is the same for all observers, show that Alice and Denise give different answers to this question.

5 Lorentz Transformation

In problem 3.2 we found the formulae which describe what is called a **Gallilean Transformation**. This describes how we go back and forth between two reference frames according to Classical Mechanics. We found the formulae to be

$$t' = t \qquad x' = x - tv$$

In Special Relativity, the formulae which explain how you go back and forth between two reference frames are called a **Lorentz Transformation**. These formulae tell you how a moving observer thinks time and space are changing (the t' and x' coordinates) as compared to a stationary observer (the t and x coordinates). In the context of problem 3.2, the relationship between t' (how much time Barbara thinks has pased) and t (how much time Alice thinks has passed) is given by

$$t' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(t - \frac{v}{c}x\right)$$

The relationship between x' (how Barbara perceives distance) and x (how Alice perceives distance) is given by

$$x' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(x - \frac{v}{c}t\right)$$

Let's try to get used to working with these formulae.

5.1 We return to the scenario discussed in section 3 where Alice is floating through space and Barbara passes her at $v = 0.5c$. Fill in the table below with the change of coordinates as given by the Lorentz Transformation.

How does this chart compare to that in 3.2(c)?

5.2 Repeat problem 5.1 using the speeds from problem 2.1. What do you notice when the speeds are close to c? What about when they are much smaller than c?